

STANDARD PARABOLIC SUBSETS OF HIGHEST WEIGHT MODULES

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ABSTRACT. In this paper we study certain fundamental and distinguished subsets of weights of an arbitrary highest weight module over a complex semisimple Lie algebra. These sets $\text{wt}_J \mathbb{V}^\lambda$ are defined for each highest weight module \mathbb{V}^λ and each subset J of simple roots; we term them “standard parabolic subsets of weights”. It is shown that for any highest weight module, the sets of simple roots whose corresponding standard parabolic subsets of weights are equal form intervals in the poset of subsets of the set of simple roots under containment. Moreover, we provide closed-form expressions for the maximum and minimum elements of the aforementioned intervals for all highest weight modules \mathbb{V}^λ over semisimple Lie algebras \mathfrak{g} . Surprisingly, these formulas only require the Dynkin diagram of \mathfrak{g} and the integrability data of \mathbb{V}^λ . As a consequence, we extend classical work by Satake, Borel–Tits, Vinberg, and Casselman, as well as recent variants by Cellini–Marietti to all highest weight modules.

We further compute the dimension, stabilizer, and vertex set of standard parabolic faces of highest weight modules, and show that they are completely determined by the aforementioned closed-form expressions. We also compute the f -polynomial and a minimal half-space representation of the convex hull of the set of weights. These results were recently shown for the adjoint representation of a simple Lie algebra, but analogues remain unknown for any other finite- or infinite-dimensional highest weight module. Our analysis is uniform and type-free, across all semisimple Lie algebras and for arbitrary highest weight modules.

1. INTRODUCTION

This paper continues the analysis of arbitrary highest weight modules over a complex semisimple Lie algebra \mathfrak{g} , which was initiated in [Kh]. Highest weight modules are fundamental in the study of Lie algebras and representation theory. Classically, they are crucial in studying the (parabolic) Bernstein–Gelfand–Gelfand Category \mathcal{O} , primitive ideals of $U(\mathfrak{g})$, quantum groups and crystals (see e.g. [Dix, HK, Hu]). More recently, highest weight modules and certain distinguished subsets of their weights have gained renewed attention for several reasons, including the study of Kirillov–Reshetikhin modules over quantum affine Lie algebras, abelian ideals, categorification, weight multiplicities, and the combinatorics of root and (pseudo-)Weyl polytopes. (See e.g. [CM, CP, CG, Ka1, Ka2, Me], and the references therein and also in [Kh].) Yet while certain special families such as (parabolic)

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Verma modules and their finite-dimensional quotients are well-understood (but not fully so), the same is not true of arbitrary highest weight modules.

The goal of the present paper – as of previous work [Kh] – is to improve our understanding of general highest weight modules \mathbb{V}^λ . (All notation is explained in Section 2.) In [Kh], the notion of the Weyl polytope was extended, from finite-dimensional simple modules to general highest weight modules. More precisely, we showed that for a large class of highest weight modules \mathbb{V}^λ with highest weight $\lambda \in \mathfrak{h}^*$, the convex hull of the weights is a convex polyhedron that is invariant under a distinguished parabolic subgroup of the Weyl group. Moreover, for certain highest weight modules \mathbb{V}^λ – including all simple highest weight modules $L(\lambda)$ for $\lambda \in \mathfrak{h}^*$ – we computed the set of weights, in three different ways.

A question that has been studied in detail in the literature by Satake, Borel–Tits, Vinberg, Casselman, Cellini–Marietti, and others (see [Kh, Section 2.4]), involves analyzing inclusion relations between pairs of faces of the aforementioned polyhedron for finite-dimensional highest weight modules, i.e., for the family of Weyl polytopes. However, the redundancies in this analysis have not been fully classified. Similar results for general highest weight modules remain even more elusive. In the first main result of this paper (see Theorem A in Section 3), we show that to each “standard parabolic face” correspond certain unique minimum and maximum sets of simple roots. Moreover, we provide closed-form expressions for these extremal subsets of simple roots, in the process completely classifying the aforementioned redundancies, for arbitrary highest weight modules.

Note that neither the existence of these extremal subsets of roots, nor closed-form formulas for them, were known even for the thoroughly studied and well-understood family of simple, finite-dimensional \mathfrak{g} -modules, excepting the adjoint representation. In contrast, our formulas hold for arbitrary highest weight modules and arbitrary faces, and are type-free. Moreover, these formulas can be deduced directly from the root data of the Dynkin diagram of \mathfrak{g} and the set of “integrable simple roots” for the highest weight module. In particular, the present paper completely resolves, for all modules \mathbb{V}^λ , the aforementioned problem studied by Vinberg [Vi] and others for finite-dimensional \mathbb{V}^λ . We also extend recent work [CM], which discussed the adjoint representation for simple \mathfrak{g} .

Another goal is to compute convexity-theoretic quantities related to highest weight modules. For instance, it is natural to ask what is the dimension, or the stabilizer subgroup in the Weyl group, of a given face of the convex hull of the set of weights. Moreover, if this hull is a convex polyhedron, is it possible to compute its f -polynomial, or a minimal half-space representation? Once again, the answers are not known in the literature except for the adjoint representation for simple \mathfrak{g} . Our second objective in this paper is to answer all of these questions for general highest weight modules \mathbb{V}^λ . We show in this paper that not only Weyl polytopes/finite-dimensional simple modules, but in fact all highest weight modules over semisimple \mathfrak{g} , are amenable to detailed analysis via representation theory and convex analysis. See Theorem B in Section 3 for more details.

A related motivation involves the recent interesting work [CM] of Cellini and Marietti. In [CM], the authors studied the combinatorics of irreducible root systems in great detail, and established several interesting properties of the root polytope in terms of the corresponding affine root system. (Recall that the root polytope is defined for a simple Lie algebra \mathfrak{g} as the convex hull of the set of roots, and it

coincides with the Weyl polytope for the highest root.) It is natural to ask if the results in [CM] are specific manifestations of broader phenomena that hold for all Weyl polytopes, or even for the convex hulls of weights of arbitrary highest weight modules \mathbb{V}^λ (which naturally extend the notion of the Weyl polytope, by results in [Kh]). Formulating and proving such results would result in a deeper understanding of Weyl polytopes, finite-dimensional \mathfrak{g} -modules, and more. However, we were unable to find such results in the literature other than for the adjoint representation for simple \mathfrak{g} (in [CM]). Indeed, the situation for semisimple \mathfrak{g} and general $\lambda, \mathbb{V}^\lambda$ is far more involved than that for simple \mathfrak{g} and the adjoint representation, or even for finite-dimensional modules. There are various technical reasons why the case of general \mathbb{V}^λ is much harder, as we explain below (see Remark 3.9). Nevertheless, the present paper completely settles many of the questions addressed in [CM], by proving them for every module \mathbb{V}^λ .

A special case of our main theorems is that Cellini–Marietti’s results hold with the exact same formulas, for all finite-dimensional \mathbb{V}^λ with λ having the same support as the affine root of (simple) \mathfrak{g} . Thus we immediately obtain hitherto unknown results about a large family of Weyl polytopes, with proofs coming from representation theory as opposed to the combinatorics of root systems.

Organization. This paper is organized as follows. Section 2 reviews basic notation as well as previous results in the literature, which lead to several natural and motivating questions. We then present our three main results in Section 3. Each of the following three sections is devoted to proving one of these main theorems. These results discuss inclusion relations between standard parabolic subsets of weights of highest weight modules, as well as computing dimensions of affine hulls, stabilizers in parabolic Weyl subgroups, f -polynomials, and minimal half-space representations in arbitrary highest weight \mathfrak{g} -modules. In Section 7, we show that the long roots, as well as the highest and lowest roots among them, which are standard notions in the adjoint representation, have natural analogues in compact standard parabolic faces of all highest weight modules. Finally in Section 8, we provide a dictionary to explain how our work specializes to the results in recent work [CM] on the root polytope for a simple Lie algebra, in terms of the corresponding affine root system. We further show how the results in [CM] in fact hold for a large family of Weyl polytopes.

2. NOTATION AND PRELIMINARY RESULTS

We begin by setting some notation. Given an \mathbb{R} -vector space \mathbb{V} and $R \subset \mathbb{R}$, $X, Y \subset \mathbb{V}$, define: (a) $R_+ := R \cap [0, \infty)$, (b) $X \pm Y$ to be the Minkowski sum and difference of X, Y , (c) $\text{conv}_{\mathbb{R}} X$ to denote the convex hull of X ; and (d) let RX denote the set of all finite \mathbb{R} -linear combinations of elements of X with coefficients in R .

Fix a complex semisimple Lie algebra \mathfrak{g} as well as a triangular decomposition $\mathfrak{g} = \mathfrak{n}^+ \oplus \mathfrak{h} \oplus \mathfrak{n}^-$. Let the corresponding root system be Φ . Corresponding to \mathfrak{n}^+ , fix distinguished \mathbb{C} -bases of simple roots $\Delta := \{\alpha_i : i \in I\} \subset \Phi$ and the associated fundamental weights $\Omega := \{\omega_i : i \in I\}$, both indexed by I . For any $J \subset I$, define $\Delta_J := \{\alpha_j : j \in J\}$, and Ω_J similarly. Let $\mathfrak{h}_{\mathbb{R}}^* := \mathbb{R}\Delta$ be the real form of \mathfrak{h}^* ; then $\mathfrak{h}_{\mathbb{R}}^* = \mathbb{R}\Omega = \mathbb{R}\Omega_I$. Moreover, \mathfrak{h}^* has a standard partial order via: $\lambda \geq \mu$ if $\lambda - \mu \in \mathbb{Z}_+ \Delta$. Next, define $P := \mathbb{Z}\Omega \supset Q := \mathbb{Z}\Delta$ to be the weight and root lattices

in $\mathfrak{h}_{\mathbb{R}}^*$ respectively, and

$$(2.1) \quad \begin{aligned} P_J^+ &:= \mathbb{Z}_+ \Omega_J, & Q_J^+ &:= \mathbb{Z}_+ \Delta_J, & P^+ &:= P_I^+, & Q^+ &:= Q_I^+, \\ \Phi_J^\pm &:= \Phi \cap \pm Q_J^\pm, & \Phi^\pm &:= \Phi_I^\pm. \end{aligned}$$

Thus, $P^+ = P_I^+$ is the set of dominant integral weights. Let $(,)$ be the positive definite symmetric bilinear form on $\mathfrak{h}_{\mathbb{R}}^*$ induced by the restriction of the Killing form on \mathfrak{g} to $\mathfrak{h}_{\mathbb{R}}$. Then $(\omega_i, \alpha_j) = \delta_{i,j}(\alpha_j, \alpha_j)/2 \forall i, j \in I$. Also define h_i to be the unique element of \mathfrak{h} identified with $(2/(\alpha_i, \alpha_i))\alpha_i$ via the Killing form. The elements h_i form an \mathbb{R} -basis of $\mathfrak{h}_{\mathbb{R}}$. Fix a set of Chevalley generators $\{x_{\alpha_i}^\pm \in \mathfrak{n}^\pm : i \in I\}$ such that $[x_{\alpha_i}^+, x_{\alpha_j}^-] = \delta_{ij}h_i$ for all $i, j \in I$. Also extend $(,)$ to all of \mathfrak{h}^* . Finally, the Weyl group is the finite subgroup $W \subset O(\mathfrak{h}^*)$ generated by the simple reflections $\{s_i = s_{\alpha_i} : i \in I\}$, where s_i sends a weight $\lambda \in \mathfrak{h}^*$ to $\lambda - \lambda(h_i)\alpha_i$. Now define W_J to be the subgroup of W generated by $\{s_j : j \in J\}$, with unique longest element w_\circ^J .

We now discuss various distinguished highest weight modules. Given $\lambda \in \mathfrak{h}^*$, define $M(\lambda)$ to be the Verma module with highest weight λ , and $L(\lambda)$ to be its unique simple quotient. Thus $M(\lambda) := U\mathfrak{g}/U\mathfrak{g}(\mathfrak{n}^+ + \ker \lambda)$. A highest weight module is a quotient of some Verma module, and is usually denoted in this paper by $M(\lambda) \twoheadrightarrow \mathbb{V}^\lambda$. Note that \mathbb{V}^λ is finite-dimensional if and only if $\lambda \in P^+$ is dominant integral and $\mathbb{V}^\lambda = L(\lambda)$ is simple. In this case the convex hull of the weights $\text{wt } L(\lambda)$ is a compact polytope, called a *Weyl polytope* and denoted by $\mathcal{P}(\lambda) := \text{conv}_{\mathbb{R}} \text{wt } L(\lambda)$.

Given $\lambda \in \mathfrak{h}^*$, define $J_\lambda := \{i \in I : \lambda(h_i) \in \mathbb{Z}_+\}$. Let \mathfrak{g}_J denote the semisimple Lie subalgebra of \mathfrak{g} generated by $\{x_{\alpha_j}^\pm : j \in J\}$, and define the parabolic Lie subalgebra $\mathfrak{p}_J := \mathfrak{g}_J + \mathfrak{h} + \mathfrak{n}^+$ for all $J \subset I$. Now given $\lambda \in \mathfrak{h}^*$ and $J \subset J_\lambda$, define the (J) -parabolic Verma module with highest weight λ to be $M(\lambda, J) := U(\mathfrak{g}) \otimes_{U(\mathfrak{p}_J)} L_J(\lambda)$. Here, $L_J(\lambda)$ is a simple finite-dimensional highest weight module over the Levi subalgebra $\mathfrak{h} + \mathfrak{g}_J$; it is also killed by $\mathfrak{g}_{I \setminus J} \cap \mathfrak{n}^+$ (in $M(\lambda, J)$). Note that parabolic Verma modules include all Verma modules (for which $J = \emptyset$) and all finite-dimensional simple modules (for which $\lambda \in P^+$ and $J = I$).

The following definition introduces the principal objects of study in the paper. As we presently discuss, these objects have been studied in the literature for finite-dimensional \mathfrak{g} -modules, most intensively for the adjoint representation.

Definition 2.2. Given $\lambda \in \mathfrak{h}^*$ and $M(\lambda) \twoheadrightarrow \mathbb{V}^\lambda$, a *standard parabolic (sub)set of weights of $\text{wt } \mathbb{V}^\lambda$* is $\text{wt}_J \mathbb{V}^\lambda := (\lambda - \mathbb{Z}_+ \Delta_J) \cap \text{wt } \mathbb{V}^\lambda$ for some $J \subset I$; and a *standard parabolic face* is $\text{conv}_{\mathbb{R}} \text{wt}_J \mathbb{V}^\lambda$.

We now recall previous work involving standard parabolic subsets and faces for various classes of highest weight modules $M(\lambda) \twoheadrightarrow \mathbb{V}^\lambda$. To our knowledge, these results were previously known for parabolic Verma modules, but not for other highest weight modules. The first result establishes integrability for a unique distinguished subset of simple roots, for arbitrary modules \mathbb{V}^λ .

Theorem 2.3 (Khare, [Kh, Theorem A]). *Given $\lambda \in \mathfrak{h}^*$ and $M(\lambda) \twoheadrightarrow \mathbb{V}^\lambda$, there exists a unique subset $J(\mathbb{V}^\lambda) \subset I$ such that the following are equivalent: (a) $J \subset J(\mathbb{V}^\lambda)$; (b) $\text{wt}_J \mathbb{V}^\lambda$ is finite; (c) $\text{wt}_J \mathbb{V}^\lambda$ is W_J -stable; (d) $\text{wt } \mathbb{V}^\lambda$ is W_J -stable. Moreover, if \mathbb{V}^λ is spanned by v_λ , then*

$$(2.4) \quad J(\mathbb{V}^\lambda) := \{i \in J_\lambda : (x_{\alpha_i}^-)^{\lambda(h_i)+1} v_\lambda = 0\}.$$

In particular, if \mathbb{V}^λ is a parabolic Verma module $M(\lambda, J')$ for $J' \subset J_\lambda$, or a simple module $L(\lambda)$, then $J(\mathbb{V}^\lambda) = J'$ or J_λ respectively.

The subset $J(\mathbb{V}^\lambda)$ is ubiquitous and crucially used throughout the remainder of the paper.

The next result establishes for a large class of highest weight modules \mathbb{V}^λ that the convex hull of the set of weights is a polyhedron. It is also possible to compute the vertices, faces, and stabilizer subgroup in W of this polyhedron.

Theorem 2.5 (Khare, [Kh, Theorems B and C]). *Suppose $(\lambda, \mathbb{V}^\lambda)$ satisfy one of the following: (a) $\lambda(h_i) \neq 0 \forall i \in I$ and \mathbb{V}^λ is arbitrary; (b) $|J_\lambda \setminus J(\mathbb{V}^\lambda)| \leq 1$ (e.g., if \mathbb{V}^λ is simple for any $\lambda \in \mathfrak{h}^*$); (c) $\mathbb{V}^\lambda = M(\lambda, J')$ for some $J' \subset J_\lambda$; or (d) \mathbb{V}^λ is pure (in the sense of [Fe]).*

Then the convex hull (in Euclidean space) $\text{conv}_{\mathbb{R}} \text{wt } \mathbb{V}^\lambda \subset \lambda + \mathfrak{h}_{\mathbb{R}}^$ is a convex polyhedron with vertices $W_{J(\mathbb{V}^\lambda)}(\lambda)$, and the stabilizer subgroup in W of both $\text{wt } \mathbb{V}^\lambda$ and $\text{conv}_{\mathbb{R}} \text{wt } \mathbb{V}^\lambda$ is $W_{J(\mathbb{V}^\lambda)}$. Moreover, a nonempty subset $Y \subset \text{wt } \mathbb{V}^\lambda$ maximizes a linear functional $\varphi \in \mathfrak{h}$ (i.e., Y is the set of weights on a supporting hyperplane) if and only if $Y = w(\text{wt}_J \mathbb{V}^\lambda)$ for some $w \in W_{J(\mathbb{V}^\lambda)}$ and $J \subset I$.*

Theorem 2.5 extends the notion of the Weyl polytope from finite-dimensional modules $L(\lambda)$ to general highest weight modules \mathbb{V}^λ . The result also extends the classification by Chari *et al* [CDR] of all maximizer subsets in the set of roots (i.e., in $\text{wt } \mathfrak{g}$), as well as previous work [KR] on maximizer subsets in the weights of parabolic Verma modules. Such results were used by Chari and her coauthors in [CG, CKR] to study distinguished categories and associated families of Koszul algebras arising from Kirillov–Reshetikhin modules over quantum affine Lie algebras.

We now discuss the motivations behind the present paper. Theorem 2.5 and previous work in [Vi, KR] shows that the sets $\text{wt}_J \mathbb{V}^\lambda$ form a distinguished family of subsets of weights for a very large class of highest weight modules \mathbb{V}^λ , including all parabolic Verma modules $M(\lambda, J')$ and their simple quotients. In particular, the standard parabolic subsets $\text{wt}_J \mathbb{V}^\lambda$ and their $W_{J(\mathbb{V}^\lambda)}$ -conjugates are precisely the sets of weights on the faces of the convex hull $\text{conv}_{\mathbb{R}} \text{wt } \mathbb{V}^\lambda$. Equivalently, standard parabolic subsets and their Weyl group conjugates are precisely the maximizer subsets of weights in $\text{wt } \mathbb{V}^\lambda$. More precisely, if $\mathbb{V}_\lambda^\lambda = \mathbb{C}v_\lambda$ and $\rho_J := \sum_{j \in J} \omega_j$, then

$$(2.6) \quad \text{wt}_J \mathbb{V}^\lambda = \text{wt } \mathbb{V}^\lambda \cap (\lambda - \mathbb{Z}\Delta_J) = \text{wt } U(\mathfrak{g}_J)v_\lambda = \arg \max_{\text{wt } \mathbb{V}^\lambda} (\rho_{I \setminus J}, -).$$

Also note by [Kh, Theorem C] that the sets $\text{wt}_J \mathbb{V}^\lambda$ are precisely the *weak \mathbb{A} -faces* of $\text{wt } \mathbb{V}^\lambda$ for “most” \mathbb{V}^λ and all additive subgroups $\mathbb{A} \subset (\mathbb{R}, +)$. Thus, the goal of this paper is to study standard parabolic sets of weights in detail, for arbitrary highest weight \mathfrak{g} -modules \mathbb{V}^λ .

Given a finite-dimensional simple module, or more generally a parabolic Verma module, it is natural to classify its faces and the redundancies among them. Results along these lines were shown by Satake, Borel–Tits, Vinberg, and Casselman for Weyl polytopes $\text{conv}_{\mathbb{R}} \text{wt } M(\lambda, I)$; see [Kh, Section 2.4] for more details. Here we state Vinberg’s formulation using the present notation.

Theorem 2.7 (Vinberg, [Vi]). *Suppose $\lambda \in P^+$. Then $\text{conv}_{\mathbb{R}} \text{wt } L(\lambda)$ is a W -invariant convex polytope with vertex set $W(\lambda)$. Every face of this polytope is W -conjugate to a unique standard parabolic face $\text{conv}_{\mathbb{R}} \text{wt}_J L(\lambda)$.*

The property of interest here is the “uniqueness” of the standard parabolic face (since the remainder of the result is already known for general modules \mathbb{V}^λ by Theorem 2.5).

In an interesting recent paper [CM], Cellini and Marietti studied in great detail, the *root polytope* $\text{conv}_{\mathbb{R}} \text{wt } \mathfrak{g}$ of a simple Lie algebra \mathfrak{g} and the set of integral weights contained in it. The authors showed several hitherto unexplored combinatorial properties of $\text{wt } \mathfrak{g} = \Phi \sqcup \{0\}$. In particular, they classified redundancies between the faces of the root polytope $\text{conv}_{\mathbb{R}} \text{wt } \mathfrak{g}$:

Theorem 2.8 (Cellini–Marietti, [CM, Theorem 1.2]). *Suppose \mathfrak{g} is simple, with highest root $\theta \in \Phi^+$. Given $\theta = \sum_{i \in I} m_i \alpha_i$ and $J \subset I$, define $F_J \subset \mathbb{R}\Delta$ to be the convex hull of the roots $\mu \in \Phi$ such that $(\mu, \omega_j) = m_j$ for all $j \in J$. Then there exist subsets $\partial J, \overline{J} \subset I$ such that for $J' \subset I$,*

$$F_{J'} = F_J \iff \partial J \subset J' \subset \overline{J}.$$

In other words, the set of possible $J' \subset I$ such that $\text{wt}_{J'} \mathfrak{g} = \text{wt}_J \mathfrak{g}$ forms an interval in the poset of subsets of I under containment. Note also that $F_J = \text{conv}_{\mathbb{R}} \text{wt}_{I \setminus J} \mathfrak{g}$ for all $J \subset I$. Thus Theorem 2.8 says that for all J , there exist smallest and largest sets J_{\min} and J_{\max} , such that $\{J' : \text{conv}_{\mathbb{R}} \text{wt}_{J'} \mathfrak{g} = \text{conv}_{\mathbb{R}} \text{wt}_J \mathfrak{g}\} = [J_{\min}, J_{\max}]$.

The present discussion, including Theorems 2.7 and 2.8, can be reformulated as follows. Using Theorem 2.3, define the *face map* to be

$$(2.9) \quad \mathcal{F}_{\mathbb{V}^\lambda} : W_{J(\mathbb{V}^\lambda)} \times 2^I \rightarrow 2^{\text{wt } \mathbb{V}^\lambda}, \quad \mathcal{F}_{\mathbb{V}^\lambda}(w, J) := w(\text{wt}_J \mathbb{V}^\lambda),$$

where 2^S denotes the power set of a set S . We term $\mathcal{F}_{\mathbb{V}^\lambda}$ the face map because by (2.6), $\mathcal{F}_{\mathbb{V}^\lambda}(w, J)$ equals the set of weights on a face of $\text{conv}_{\mathbb{R}} \text{wt } \mathbb{V}^\lambda$ for all $w \in W_{J(\mathbb{V}^\lambda)}$ and $J \subset I$. Next, define

$$(2.10) \quad \mathcal{F}_{\mathbb{V}^\lambda}^{(1)} = \mathcal{F}_{\mathbb{V}^\lambda}(1, -) : 2^I \rightarrow 2^{\text{wt } \mathbb{V}^\lambda}, \quad \mathcal{F}_{\mathbb{V}^\lambda}^{(1)}(J) := \text{wt}_J \mathbb{V}^\lambda.$$

(Note that both $\mathcal{F}_{\mathbb{V}^\lambda}$ and $\mathcal{F}_{\mathbb{V}^\lambda}^{(1)}$ have finite sets as domains.) Now Theorem 2.7 shows a connection between the fibers of $\mathcal{F}_{L(\lambda)}$ and of $\mathcal{F}_{L(\lambda)}^{(1)}$: namely, if $\lambda \in P^+$ and $\mathcal{F}_{L(\lambda)}(w, J) = \mathcal{F}_{L(\lambda)}(w', J')$, then $\mathcal{F}_{L(\lambda)}^{(1)}(J) = \mathcal{F}_{L(\lambda)}^{(1)}(J')$. On the other hand, Theorem 2.8 classifies the fibers of $\mathcal{F}_{\mathfrak{g}}^{(1)}$ and shows that they are intervals of the form $[J_{\min}, J_{\max}]$. Note that a similar result holds trivially for arbitrary Verma modules $\mathbb{V}^\lambda = M(\lambda)$, since the assignment $J \mapsto \text{wt}_J M(\lambda)$ is one-to-one. In previous work [Kh] we also showed a similar result for all highest weight modules with “generic” highest weight:

Theorem 2.11 (Khare, [Kh, Corollary 3.16]). *Suppose \mathfrak{g} is semisimple, $\lambda(h_i) \neq 0 \forall i \in I$, and $M(\lambda) \twoheadrightarrow \mathbb{V}^\lambda$ is arbitrary. Then $J \mapsto \text{wt}_J \mathbb{V}^\lambda$ is an injective map from subsets of I to subsets of $\text{wt } \mathbb{V}^\lambda$.*

In other words, if λ is not on any simple root hyperplane, then $\mathcal{F}_{\mathbb{V}^\lambda}^{(1)}$ is injective on 2^I . (Thus it has fiber $[J, J]$ at each subset J .)

The following questions naturally arise now: (1) In what level of generality do the above results hold? In other words, for which highest weight modules \mathbb{V}^λ can the fibers of $\mathcal{F}_{\mathbb{V}^\lambda}$ and $\mathcal{F}_{\mathbb{V}^\lambda}^{(1)}$ be related and/or classified? (2) If the analogue of Theorem 2.8 holds for a highest weight module \mathbb{V}^λ (i.e., the fibers of $\mathcal{F}_{\mathbb{V}^\lambda}^{(1)}$ are all intervals), then is there a closed-form expression for the sets J_{\min}, J_{\max} ? (3) Can

the redundancies among the set of faces $\{w(\text{wt}_J \mathbb{V}^\lambda) : w \in W_{J(\mathbb{V}^\lambda)}, J \subset I\}$ – i.e., the fibers of the map $\mathcal{F}_{\mathbb{V}^\lambda}$ – be classified for all highest weight modules \mathbb{V}^λ ? Answers to the second question are known only for the adjoint representation for simple \mathfrak{g} (and now when λ avoids the simple root hyperplanes, by our result, Theorem 2.11). Neither the existence of J_{\min}, J_{\max} , nor formulas for them, are known for other finite-dimensional modules $M(\lambda, I) = L(\lambda)$. Formulas in the case of general modules \mathbb{V}^λ are even harder, as are the first and third questions.

Thus, one of the main accomplishments of this paper is to provide positive answers to all of the above questions in complete generality, for all highest weight modules \mathbb{V}^λ and all semisimple \mathfrak{g} . Remarkably, the formulas for J_{\min} and J_{\max} can be read off from the Dynkin diagram of \mathfrak{g} , using only the data of the sets J and $J(\mathbb{V}^\lambda)$ as well as the support set of λ . See Theorem A for a precise formulation. We also completely classify the redundancies between faces of arbitrary highest weight modules \mathbb{V}^λ ; i.e., we compute the fibers of the face map $\mathcal{F}_{\mathbb{V}^\lambda}$. See Proposition 3.10.

Our second motivation comes from further combinatorial results shown recently by Cellini and Marietti [CM] for the root system of a simple Lie algebra \mathfrak{g} . In [CM] the authors compute the dimension and stabilizer subgroup of the faces of the root polytope of \mathfrak{g} .

Theorem 2.12 (Cellini–Marietti, [CM, Section 1]). *Suppose \mathfrak{g} is simple, with highest root θ . Given $\lambda = \theta$ and $J \subset I$, define $\partial J, \overline{J}, F_J$ as in Theorem 2.8. Then F_J has dimension $|I| - |\overline{J}|$ and stabilizer subgroup $W_{I \setminus \partial J}$ in W , and its barycenter lies in $\mathbb{R}_+ \Omega_{\partial J}$. Moreover, F_J has $[W_{I \setminus J} : W_{(I \setminus J) \cap \{\theta\}^\perp}]$ vertices, where $\{\theta\}^\perp$ denotes the set of simple roots orthogonal to θ .*

It is natural to ask if similar results hold for Weyl polytopes, or more generally for all modules \mathbb{V}^λ . The present paper provides positive answers for general highest weight modules \mathbb{V}^λ over all semisimple \mathfrak{g} ; see Theorem B as well as Lemma 5.2.

3. MAIN RESULTS

In this section we present the main results of the paper. The following notation is required to state and prove these results.

Definition 3.1. Suppose $\lambda \in \mathfrak{h}^*$, $M(\lambda) \twoheadrightarrow \mathbb{V}^\lambda$, and $J \subset I$.

- (1) Define $\pi_J : \mathfrak{h}^* = \mathbb{C}\Omega_I \twoheadrightarrow \mathbb{C}\Omega_J$ to be the projection map with kernel $\mathbb{C}\Omega_{I \setminus J}$.
- (2) Denote the support of $\lambda \in \mathfrak{h}^*$ by $\text{supp}(\lambda) := \{i \in I : (\lambda, \alpha_i) \neq 0\} = \{i \in I : \pi_{\{i\}}(\lambda) \neq 0\}$. Thus $\lambda \in \mathbb{C}\Omega_{\text{supp}(\lambda)} \forall \lambda \in \mathfrak{h}^*$.
- (3) If $J \subset J(\mathbb{V}^\lambda)$, define J_{\min} to be the union of the connected components C in the Dynkin diagram of J such that $\pi_C(\lambda) \neq 0$, i.e., such that $C \cap \text{supp}(\lambda)$ is nonempty.
- (4) Given $J \subset I$ and \mathbb{V}^λ , partition I as $I = \bigsqcup_{i=1}^6 J_i(\mathbb{V}^\lambda)$, where

$$J_1(\mathbb{V}^\lambda) := I \setminus (J \cup J(\mathbb{V}^\lambda)), \quad J_2(\mathbb{V}^\lambda) := J \setminus J(\mathbb{V}^\lambda), \quad J_3(\mathbb{V}^\lambda) := (J \cap J(\mathbb{V}^\lambda))_{\min},$$

$$J_5(\mathbb{V}^\lambda) := (J \cap J(\mathbb{V}^\lambda)) \setminus (J_3(\mathbb{V}^\lambda) \sqcup J_4(\mathbb{V}^\lambda)), \quad J_6(\mathbb{V}^\lambda) := J(\mathbb{V}^\lambda) \setminus J,$$

and $J_4(\mathbb{V}^\lambda)$ is the union of those connected components C of the Dynkin diagram of $J \cap J(\mathbb{V}^\lambda)$, for which $\pi_C(\lambda) = 0$ but $\Delta_C \not\subseteq \Delta_{J_2(\mathbb{V}^\lambda)}$.

- (5) Define for any $X \subset \mathfrak{h}^*$, the set $X^\perp := \{i \in I : (\alpha_i, x) = 0 \forall x \in X\}$. Now define $J^\perp := \Delta_{J^\perp}$.

We provide some elaboration on the sets $J_i(\mathbb{V}^\lambda)$. These explanations are helpful in building some intuition about the sets $J_i(\mathbb{V}^\lambda)$; all reasoning is provided below, in the proof of Theorem A.

- $J_1(\mathbb{V}^\lambda)$ does not play any role in the set $\text{wt}_J \mathbb{V}^\lambda$ for a highest weight module \mathbb{V}^λ .
- $J_2(\mathbb{V}^\lambda)$ consists of the indices $j \in J$ such that $\lambda - \mathbb{Z}_+ \alpha_j \subset \text{wt}_J \mathbb{V}^\lambda$.
- For $l = 3, 4, 5$, $J_l(\mathbb{V}^\lambda)$ consists of the graph components C of the Dynkin diagram of $J \cap J(\mathbb{V}^\lambda)$, for which: (a) $\pi_C(\lambda) \neq 0$, if $l = 3$; (b) $\pi_C(\lambda) = 0$ but $(\Delta_{J \setminus J(\mathbb{V}^\lambda)}, \Delta_C) \neq 0$, if $l = 4$; and (c) $\pi_C(\lambda) = 0 = (\Delta_{J \setminus J(\mathbb{V}^\lambda)}, \Delta_C)$, if $l = 5$.

We now state the main results of this paper. Our first result achieves three goals: first, it provides a complete characterization of when two standard parabolic sets of weights coincide, for all highest weight modules \mathbb{V}^λ . (As explained in Remark 8.7, this allows for the complete classification of inclusion relations between standard parabolic subsets/faces.) Next, the result also demonstrates the existence of the sets J_{\min}, J_{\max} for all \mathbb{V}^λ . Finally, it provides the first closed-form expressions for the sets J_{\min}, J_{\max} for any highest weight module over semisimple \mathfrak{g} (with the sole exception of $\mathbb{V}^\lambda = \mathfrak{g}$ for simple \mathfrak{g} , for which a different pair of formulae were proved very recently in [CM]).

Theorem A. *Suppose $\lambda \in \mathfrak{h}^*$, $M(\lambda) \twoheadrightarrow \mathbb{V}^\lambda$, \mathbb{V}^λ is spanned by v_λ , and $J \subset I$. Define subsets $J_{\min}, J_{\max} \subset J(\mathbb{V}^\lambda)$ as follows (notation as in Definition 3.1):*

(3.2)

$$J_{\min} = J_3(\mathbb{V}^\lambda) \sqcup J_4(\mathbb{V}^\lambda) = \bigcup_{\mu \in \{0\} \sqcup \Delta_{J \setminus J(\mathbb{V}^\lambda)}} (J \cap J(\mathbb{V}^\lambda))_3(L_{J(\mathbb{V}^\lambda)}(\pi_{J(\mathbb{V}^\lambda)}(\lambda - \mu))),$$

(3.3)

$$\begin{aligned} J_{\max} &= (J \cap J(\mathbb{V}^\lambda)) \sqcup [J_6(\mathbb{V}^\lambda) \cap \{\lambda\}^\perp \cap J_{\min}^\perp \cap J_2(\mathbb{V}^\lambda)^\perp] \\ &= J_{\min} \sqcup (J(\mathbb{V}^\lambda) \cap \{\lambda\}^\perp \cap J_{\min}^\perp \cap J_2(\mathbb{V}^\lambda)^\perp). \end{aligned}$$

Then all equalities in (3.2) and (3.3) are valid. Moreover, the following are equivalent for $J' \subset I$:

- (1) There exist $w, w' \in W_{J(\mathbb{V}^\lambda)}$ such that $w(\text{wt}_J \mathbb{V}^\lambda) = w'(\text{wt}_{J'} \mathbb{V}^\lambda)$.
- (2) $\text{wt}_J \mathbb{V}^\lambda = \text{wt}_{J'} \mathbb{V}^\lambda$.
- (3) $\text{conv}_{\mathbb{R}} \text{wt}_J \mathbb{V}^\lambda = \text{conv}_{\mathbb{R}} \text{wt}_{J'} \mathbb{V}^\lambda$.
- (4) $U(\mathfrak{g}_J)v_\lambda = U(\mathfrak{g}_{J'})v_\lambda$.
- (5) $J \setminus J(\mathbb{V}^\lambda) = J' \setminus J(\mathbb{V}^\lambda)$ and $J_{\min} \subset J' \cap J(\mathbb{V}^\lambda) \subset J_{\max}$.

In particular, there are three formulas for J_{\min} when $J \subset J(\mathbb{V}^\lambda)$ (from Definition 3.1 and equation (3.2)), and we show in this paper that these formulas agree, as do the two proposed formulas for J_{\max} . Moreover, Theorems 2.7, 2.8, 2.11 respectively by Vinberg (and others), Cellini–Marietti, and us, are the special cases of Theorem A that were previously known. The second of these assertions holds because $F_J = \text{conv}_{\mathbb{R}} \text{wt}_{I \setminus J} \mathfrak{g}$ for simple \mathfrak{g} , as explained in Section 8 below. The third assertion holds because if $\lambda(h_i) \neq 0$ for all $i \in I$, then $J_4(\mathbb{V}^\lambda) = J_5(\mathbb{V}^\lambda) = \{\lambda\}^\perp = \emptyset$ for all \mathbb{V}^λ , whence $J_{\min} = J_{\max} = J_3(\mathbb{V}^\lambda) = J \cap J(\mathbb{V}^\lambda)$.

Further note from Theorem A that the last expression in equation (3.2) reduces the situation to the finite-dimensional setting of the “integrable top” $L_{J(\mathbb{V}^\lambda)}(\lambda - \mu)$. In fact the set corresponding to $\mu = 0$ in (3.2) is contained in the set corresponding to $\mu = \alpha_{j_2}$ for every $j_2 \in J_2(\mathbb{V}^\lambda)$, but it is nevertheless included in (3.2) (and also

in the results and proofs in the paper below) because $J_2(\mathbb{V}^\lambda)$ may be empty, e.g. in the case of finite-dimensional modules \mathbb{V}^λ .

Remark 3.4. As an aside, Theorem A answers the following branching-type question: given a finite-dimensional simple \mathfrak{g} -module $L(\lambda)$, for which subsets $J \subset I$ is its restriction to \mathfrak{g}_J still an irreducible \mathfrak{g}_J -module? By Theorem A, the answer is: all J containing I_{\min} . More generally for all $\lambda \in \mathfrak{h}^*$, the answer to the same question (for $L(\lambda)$) is: all subsets J containing $I_{\min} \sqcup (I \setminus J_\lambda)$.

Remark 3.5. It is preferable to work with arbitrary semisimple \mathfrak{g} rather than simple Lie algebras. One reason is that when $J \subset I$ is allowed to be arbitrary, disjoint connected graph components of $J \cap J(\mathbb{V}^\lambda)$ automatically force us to work with semisimple subalgebras of \mathfrak{g} .

Our next main result establishes several combinatorial facts about standard parabolic sets of weights $\text{wt}_J \mathbb{V}^\lambda$ for highest weight modules \mathbb{V}^λ .

Theorem B. *Suppose $\lambda \in \mathfrak{h}^*$, $M(\lambda) \rightarrow \mathbb{V}^\lambda$, and $J \subset I$. Then the standard parabolic face $\text{conv}_{\mathbb{R}}(\text{wt}_J \mathbb{V}^\lambda)$ has:*

- (1) *dimension $|J_{\min}| + |J \setminus J(\mathbb{V}^\lambda)|$ as well as affine hull $\lambda - \mathbb{R}\Delta_{J_{\min} \sqcup (J \setminus J(\mathbb{V}^\lambda))}$ and*
- (2) *stabilizer subgroup $W_{J_{\max}} = W_{J_{\min}} \times W_{J_{\max} \setminus J_{\min}}$ in $W_{J(\mathbb{V}^\lambda)}$.*

Now assume further that $\text{conv}_{\mathbb{R}} \text{wt } \mathbb{V}^\lambda = \text{conv}_{\mathbb{R}} \text{wt } M(\lambda, J(\mathbb{V}^\lambda))$. Then the face $\text{conv}_{\mathbb{R}} \text{wt}_J \mathbb{V}^\lambda$ has $[W_{J \cap J(\mathbb{V}^\lambda)} : W_{J \cap J(\mathbb{V}^\lambda) \cap \{\lambda\}^\perp}]$ vertices. Moreover, the polyhedron $\text{conv}_{\mathbb{R}} \text{wt } \mathbb{V}^\lambda$ has f -polynomial

$$\mathbf{f}_{\mathbb{V}^\lambda}(t) = \sum_J [W_{J(\mathbb{V}^\lambda)} : W_{J_{\max}}] t^{|J_{\min}| + |J \setminus J(\mathbb{V}^\lambda)|},$$

where we sum over the distinct elements J in the multiset $\{J_{\max} : J \subset I\}$, or in the multiset $\{J_{\min} : J \subset I\}$.

Remark 3.6. Note from Theorem B that much of the convexity-theoretic data of a highest weight module \mathbb{V}^λ is completely determined by the sets $\{J_{\min}, J_{\max} : J \subset I\}$. (This includes $I_{\max} = J(\mathbb{V}^\lambda)$.)

The assertions in Theorem B were established in the special case $\lambda = \theta$ and $\mathbb{V}^\lambda = M(\theta, I) = L(\theta) = \mathfrak{g}$ for a simple Lie algebra \mathfrak{g} , by Cellini and Marietti in their recent paper [CM] (see also [ABH] for explicit formulas for f -polynomials for all simply laced root polytopes). The assertions appear for $\mathbb{V}^\lambda = \mathfrak{g}$ in [CM] as their “main results” Theorem 1.2(1), Theorem 1.2(2), Theorem 1.1(2), and Theorem 1.3 respectively. Theorem B now shows that these results hold for a very large family of highest weight modules \mathbb{V}^λ (some of them hold for all \mathbb{V}^λ) and for all semisimple Lie algebras; see Remark 3.7. Additionally, in this paper we show several other related statements, which specialize to many of the results in [CM] in the special case $\lambda = \theta$, $\mathbb{V}^\lambda = L(\theta) = \mathfrak{g}$. See Section 8 for more details. Note that these results are not known for any other highest weight module.

Remark 3.7. We briefly remark on the assumption in the last two parts of Theorem B:

$$(3.8) \quad \text{conv}_{\mathbb{R}} \text{wt } \mathbb{V}^\lambda = \text{conv}_{\mathbb{R}} \text{wt } M(\lambda, J(\mathbb{V}^\lambda)).$$

We show in Proposition 5.1 that equation (3.8) holds under any of the four hypotheses on \mathbb{V}^λ that are listed in Theorem 2.5. Thus Theorem B holds for a very large class of highest weight modules.

The third result in this section involves obtaining a minimal half-space representation for the convex polyhedron $\text{conv}_{\mathbb{R}} \text{wt } \mathbb{V}^\lambda$. It is known for the adjoint representation (see [CM]) and is not hard to prove for Verma modules, but it is not known for any nonzero module $\mathbb{V}^\lambda \neq \mathfrak{g}$.

Theorem C. *Suppose $\lambda \in \mathfrak{h}^*$ and $M(\lambda) \twoheadrightarrow \mathbb{V}^\lambda$, and assume $\text{conv}_{\mathbb{R}} \text{wt } \mathbb{V}^\lambda = \text{conv}_{\mathbb{R}} \text{wt } M(\lambda, J(\mathbb{V}^\lambda))$. Then the convex polyhedron $\text{conv}_{\mathbb{R}} \text{wt } \mathbb{V}^\lambda$ is represented as the intersection of the half-spaces*

$$H_{i,w} := \{\mu \in \lambda - \mathbb{R}\Delta : (\lambda - \mu, w\omega_i) \geq 0\}, \quad \forall w \in W^i, i \in I,$$

where W^i is any set of coset representatives of $W_{J(\mathbb{V}^\lambda)}/W_{(I \setminus \{i\})_{\max}}$. Moreover, the representation is minimal if one runs over only the simple roots $i \in I_{\min} \sqcup (I \setminus J(\mathbb{V}^\lambda))$ such that $(I \setminus \{i\})_{\min} = I_{\min} \setminus \{i\}$.

We prove additional results in this paper, for instance involving *longest weights*, which are an analogue of long roots in the adjoint representation. We also show characterizations of when a weight is minimal in a standard parabolic subset, or when a standard parabolic face is a facet.

Remark 3.9. Note that analyzing arbitrary modules \mathbb{V}^λ and their weights is far more involved than previous work in the literature on root/Weyl polytopes. We now discuss some of the technical difficulties that arise when studying infinite-dimensional highest weight modules \mathbb{V}^λ , but do not arise for finite-dimensional modules and Weyl polytopes. One such complication is that the integrability and W -invariance properties of Weyl polytopes do not hold for $\text{conv}_{\mathbb{R}} \text{wt } \mathbb{V}^\lambda$ for infinite-dimensional modules \mathbb{V}^λ . Thus it is not readily apparent how to extend results that are known only for the adjoint representation, or even for all finite-dimensional modules, to all highest weight modules.

A second, more subtle distinction is that the set $J(\mathbb{V}^\lambda)$ equals all of I for finite-dimensional modules \mathbb{V}^λ (in fact this is a characterization of finite-dimensionality). In particular, it is easy to compute $J(\mathbb{V}^\lambda)_{\min}$ for finite-dimensional modules \mathbb{V}^λ : it equals precisely $I_{\min} = I$ if \mathfrak{g} is simple and $\lambda \neq 0$, as in previous papers [CM, CDR, CG] by Cellini, Chari, and their coauthors. (Clearly $J(\mathbb{V}^\lambda)_{\max} = I_{\max} = I$ for all semisimple \mathfrak{g} and finite-dimensional modules \mathbb{V}^λ .) In contrast, we work with all highest weight modules over an arbitrary semisimple Lie algebra. The analysis now is more delicate as one has to account for the “non-integrable directions” $I \setminus J(\mathbb{V}^\lambda)$.

We end by generalizing and completing an analysis initiated by Satake, Borel–Tits, Vinberg, and Casselman for Weyl polytopes. Recall by Theorem 2.5 that every face of a Weyl polytope, or more generally of $\text{conv}_{\mathbb{R}} \text{wt } \mathbb{V}^\lambda$ for a very large class of highest weight modules \mathbb{V}^λ , is of the form $\text{conv}_{\mathbb{R}} w(\text{wt}_J \mathbb{V}^\lambda)$ for $w \in W_{J(\mathbb{V}^\lambda)}$ and $J \subset I$. We now completely enumerate the redundancies between these faces, as well as all inclusions of standard parabolic faces, for arbitrary highest weight modules \mathbb{V}^λ .

Proposition 3.10. *Fix $\lambda \in \mathfrak{h}^*$, $M(\lambda) \twoheadrightarrow \mathbb{V}^\lambda$, and $J \subset I$. Now given $w, w' \in W_{J(\mathbb{V}^\lambda)}$ and $J' \subset I$,*

$$(3.11) \quad \begin{aligned} & \text{conv}_{\mathbb{R}} w(\text{wt}_J \mathbb{V}^\lambda) = \text{conv}_{\mathbb{R}} w'(\text{wt}_{J'} \mathbb{V}^\lambda) \iff w(\text{wt}_J \mathbb{V}^\lambda) = w'(\text{wt}_{J'} \mathbb{V}^\lambda) \\ & \iff J' \setminus J(\mathbb{V}^\lambda) = J \setminus J(\mathbb{V}^\lambda), \quad J_{\min} \subset J' \cap J(\mathbb{V}^\lambda) \subset J_{\max}, \quad w^{-1}w' \in W_{J_{\max}}. \end{aligned}$$

If $\mathbb{V}^\lambda = \mathbb{C}v_\lambda$, the following are also equivalent: (a) $\text{wt}_J \mathbb{V}^\lambda \subset \text{wt}_{J'} \mathbb{V}^\lambda$; (b) $\text{conv}_{\mathbb{R}} \text{wt}_J \mathbb{V}^\lambda \subset \text{conv}_{\mathbb{R}} \text{wt}_{J'} \mathbb{V}^\lambda$; (c) $U(\mathfrak{g}_J)v_\lambda \subset U(\mathfrak{g}_{J'})v_\lambda$; (d) $J \setminus J(\mathbb{V}^\lambda) \subset J' \setminus J(\mathbb{V}^\lambda)$ and $J_{\min} \subset J'_{\min}$.

In particular, equation (3.11) completely classifies the fibers of the maps $\mathcal{F}_{\mathbb{V}^\lambda}$ and $\mathcal{F}_{\mathbb{V}^\lambda}^{(1)}$, defined in equations (2.9), (2.10) respectively. Special cases of this result were partially known for finite-dimensional modules $L(\lambda)$, and specifically, the adjoint representation. See Theorems 2.7 and 2.8.

Proof. The first equivalences in and following (3.11) both follow from Theorem A and the fact that $\text{wt } \mathbb{V}^\lambda \cap \text{conv}_{\mathbb{R}} w(\text{wt}_J \mathbb{V}^\lambda) = w(\text{wt}_J \mathbb{V}^\lambda)$ for all $w \in W_{J(\mathbb{V}^\lambda)}$ and $J \subset I$, since $\text{wt } \mathbb{V}^\lambda$ is $W_{J(\mathbb{V}^\lambda)}$ -stable. Now $w(\text{wt}_J \mathbb{V}^\lambda) = w'(\text{wt}_{J'} \mathbb{V}^\lambda)$ if and only if $\text{wt}_J \mathbb{V}^\lambda = \text{wt}_{J'} \mathbb{V}^\lambda$ and $w^{-1}w'$ stabilizes this set. Equation (3.11) now follows by Theorems A and B. The last equivalence in the proposition now follows using Theorem B and by modifying the proof of (2) \implies (4) in Theorem A. \square

4. INCLUSION RELATIONS AMONG STANDARD PARABOLIC SUBSETS

In this section we prove Theorem A, which classifies when two standard parabolic subsets of $\text{wt } \mathbb{V}^\lambda$ are equal. (Note by Theorem 2.5 that this is also equivalent to the problem of studying inclusion relations among maximizer sets of weights, for a very large family of highest weight modules.) As the proof of Theorem A is quite involved, we begin by first studying the case where the standard parabolic subsets in question are finite. We then proceed to the general case.

4.1. Inclusion relations among finite maximizer subsets. Recall by Theorem 2.3 that the standard parabolic subsets of $\text{wt } \mathbb{V}^\lambda$ that are finite sets are all of the form $\text{wt}_J \mathbb{V}^\lambda$ with $J \subset J(\mathbb{V}^\lambda)$. We now characterize when two finite standard parabolic subsets of $\text{wt } \mathbb{V}^\lambda$ are equal.

Proposition 4.1. *Fix $\lambda \in \mathfrak{h}^*$, $M(\lambda) \twoheadrightarrow \mathbb{V}^\lambda$, and $J, J' \subset J(\mathbb{V}^\lambda)$. Then the vertices of $\text{conv}_{\mathbb{R}}(\text{wt}_J \mathbb{V}^\lambda)$ are precisely $W_J(\lambda)$. Moreover, the following are equivalent (notation in Definition 4.2):*

- (1) $\text{wt}_J \mathbb{V}^\lambda = \text{wt}_{J'} \mathbb{V}^\lambda$.
- (2) $\rho_{\text{wt}_J \mathbb{V}^\lambda} = \rho_{\text{wt}_{J'} \mathbb{V}^\lambda}$.
- (3) $\rho_{\text{wt}_J \mathbb{V}^\lambda} \in \mathbb{Q}_+ \rho_{\text{wt}_{J'} \mathbb{V}^\lambda}$.
- (4) $\pi_{J(\mathbb{V}^\lambda)}(\rho_{\text{wt}_J \mathbb{V}^\lambda}) = \pi_{J(\mathbb{V}^\lambda)}(\rho_{\text{wt}_{J'} \mathbb{V}^\lambda})$.
- (5) $\pi_{J(\mathbb{V}^\lambda)}(\rho_{\text{wt}_J \mathbb{V}^\lambda}) \in \mathbb{Q}_+ \pi_{J(\mathbb{V}^\lambda)}(\rho_{\text{wt}_{J'} \mathbb{V}^\lambda})$.
- (6) $W_J(\lambda) = W_{J'}(\lambda)$.
- (7) $\rho_{\text{wt}_J \mathbb{V}^\lambda}, \rho_{\text{wt}_{J'} \mathbb{V}^\lambda}$ are both fixed by $W_{J \cup J'}$.

This is an “intermediate” result since $J, J' \subset J(\mathbb{V}^\lambda)$. The case of general $J, J' \subset I$ is Theorem A.

The proof of Proposition 4.1 requires the following notation and results from [KR, Kh].

Definition 4.2.

- (1) Given $J \subset I$, define $\varpi_J : \lambda + \mathbb{C}\Delta_J \rightarrow \pi_J(\lambda) + \mathbb{C}\Delta_J$ (where the codomain comes from \mathfrak{g}_J) as follows: $\varpi_J(\lambda + \mu) := \pi_J(\lambda) + \mu$.
- (2) Given a nonempty finite subset $X \subset \mathfrak{h}^*$, define its average value, or barycenter, to be: $\text{avg}(X) := \frac{1}{|X|} \sum_{x \in X} x$. Also define $\rho_X := \sum_{x \in X} x$.
- (3) Given $X \subset \mathfrak{h}^*$ and $\mu \in \mathfrak{h}^*$, define the corresponding *maximizer subset* $X(\mu) := \{x \in X : (\mu, x - x') \in \mathbb{R}_+ \ \forall x' \in X\}$.

We now state various results that are repeatedly used in the present and subsequent sections, to prove the main theorems in this paper. First recall that the following special case of Proposition 4.1 has been shown in the literature, for finite-dimensional modules.

Theorem 4.3 (Khare and Ridenour, [KR, Theorem 4]). *Suppose $\lambda \in P^+$ and $J, J' \subset I = J(L(\lambda))$. The vertices of $\text{conv}_{\mathbb{R}} \text{wt}_J L(\lambda)$ are precisely $W_J(\lambda)$. Moreover, $\text{wt}_J L(\lambda) = \text{wt}_{J'} L(\lambda)$ if and only if $\rho_{\text{wt}_J L(\lambda)} = \rho_{\text{wt}_{J'} L(\lambda)}$, if and only if $W_J(\lambda) = W_{J'}(\lambda)$.*

The following result discusses how to go from the highest weight down to any other weight in $\text{wt } \mathbb{V}^\lambda$.

Lemma 4.4 ([Kh, Lemma 3.12]). *Suppose $M(\lambda) \twoheadrightarrow \mathbb{V}^\lambda$ (with highest weight space $\mathbb{C}v_\lambda$) and $\mu \in \text{wt}_J \mathbb{V}^\lambda$, for some $\lambda \in \mathfrak{h}^*$ and $J \subset I$. Then there exist $\mu_j \in \text{wt}_J \mathbb{V}^\lambda$ such that*

$$\lambda = \mu_0 > \mu_1 > \cdots > \mu_N = \mu, \quad \mu_j - \mu_{j+1} \in \Delta_J \ \forall j, \quad N \geq 0.$$

Moreover, if $\mathbb{V}^\lambda = L(\lambda)$ is simple, then so is the \mathfrak{g}_J -submodule $\mathbb{V}_J^\lambda := U(\mathfrak{g}_J)v_\lambda$.

The next result is a “transfer principle”, sending an arbitrary highest weight module \mathbb{V}^λ to its “integrable top” $\mathbb{V}_{J(\mathbb{V}^\lambda)}^\lambda$. In other words, $\varpi_J : \text{wt}_J \mathbb{V}^\lambda \rightarrow \text{wt } L_J(\pi_J(\lambda))$ is a bijection if $J \subset J(\mathbb{V}^\lambda)$.

Lemma 4.5 ([Kh, Lemma 4.3]). *Fix $\lambda \in \mathfrak{h}^*$, $M(\lambda) \twoheadrightarrow \mathbb{V}^\lambda$ generated by $0 \neq v_\lambda \in \mathbb{V}_\lambda^\lambda$, and $J \subset I$.*

- (1) $J \subset J_\lambda$ if and only if $\pi_J(\lambda) \in P^+$ (in fact, in P_J^+).
- (2) Let $\mathbb{V}_J^\lambda := U(\mathfrak{g}_J)v_\lambda$. Then for all $J, J' \subset I$, $\text{wt}_{J'} \mathbb{V}_J^\lambda = \text{wt}_{J \cap J'} \mathbb{V}^\lambda$.
- (3) \mathbb{V}_J^λ is a highest weight \mathfrak{g}_J -module with highest weight $\pi_J(\lambda)$. In other words, $M_J(\pi_J(\lambda)) \twoheadrightarrow U(\mathfrak{g}_J)v_\lambda$, where M_J denotes the corresponding Verma \mathfrak{g}_J -module.
- (4) For all $w \in W_J$ and $\mu \in \mathbb{C}\Delta_J$, $w(\varpi_J(\lambda + \mu)) = \varpi_J(w(\lambda + \mu))$.

Also recall previous results on the supports of barycenters of finite standard parabolic faces, as well as on maximizing linear functionals corresponding to standard parabolic faces.

Proposition 4.6 ([Kh, Proposition 4.10]). *Fix $\lambda \in \mathfrak{h}^*$, $M(\lambda) \twoheadrightarrow \mathbb{V}^\lambda$, and $J \subset J(\mathbb{V}^\lambda)$.*

- (1) Then $\rho_{\text{wt}_J \mathbb{V}^\lambda}$ is W_J -invariant, and in $P_{J_\lambda \setminus J}^+ \times \mathbb{C}\Omega_{I \setminus J_\lambda}$.
- (2) Define $\rho_{I \setminus J} := \sum_{i \notin J} \omega_i$. Then for all $J' \subset J_\lambda$, one has an inclusion of maximizer subsets:

$$(4.7) \quad \text{wt}_J \mathbb{V}^\lambda = (\text{wt } \mathbb{V}^\lambda)(\rho_{I \setminus J}) = (\text{wt}_{J(\mathbb{V}^\lambda)} \mathbb{V}^\lambda)(\pi_{J(\mathbb{V}^\lambda)} \rho_{\text{wt}_J \mathbb{V}^\lambda}) \subset (\text{wt } \mathbb{V}^\lambda)(\pi_{J'} \rho_{\text{wt}_J \mathbb{V}^\lambda})$$

and $0 \leq (\pi_{J'} \rho_{\text{wt}_J \mathbb{V}^\lambda})(\text{wt}_J \mathbb{V}^\lambda) \in \mathbb{Z}_+$.

Equipped with the above results, it is now possible to prove Proposition 4.1.

Proof of Proposition 4.1. The assertion about the vertices follows from Theorem 4.3 (for $\mathfrak{g}_{J(\mathbb{V}^\lambda)}$) and Lemma 4.5, via the bijection $\varpi_{J(\mathbb{V}^\lambda)}$. In the course of this reasoning, we use that $W_{J(\mathbb{V}^\lambda)}(\lambda) \subset \lambda - \mathbb{Z}_+ \Delta$, and similarly for $W_{J(\mathbb{V}^\lambda)}(\pi_{J(\mathbb{V}^\lambda)}(\lambda))$. Next, $\text{wt}_J \mathbb{V}^\lambda$ and $\text{wt}_{J'} \mathbb{V}^\lambda$ are both finite sets by Theorem 2.3. The following implications are now obvious:

$$(1) \implies (2) \implies (3) \implies (5); \quad (2) \implies (4) \implies (5).$$

Now if (5) holds, then the two (equal) weights have the same maximizer by Proposition 4.6:

$$\text{wt}_J \mathbb{V}^\lambda = (\text{wt}_{J(\mathbb{V}^\lambda)} \mathbb{V}^\lambda)(\pi_{J(\mathbb{V}^\lambda)} \rho_{\text{wt}_J \mathbb{V}^\lambda}) = (\text{wt}_{J(\mathbb{V}^\lambda)} \mathbb{V}^\lambda)(\pi_{J(\mathbb{V}^\lambda)} \rho_{\text{wt}_{J'} \mathbb{V}^\lambda}) = \text{wt}_{J'} \mathbb{V}^\lambda.$$

This proves (1) again. Now if $\text{wt}_J \mathbb{V}^\lambda = \text{wt}_{J'} \mathbb{V}^\lambda$, then their convex hulls (which are polytopes) are equal. Via $\varpi_{J(\mathbb{V}^\lambda)}$, this also means that the convex hulls of certain subsets of weights of $M := L_{J(\mathbb{V}^\lambda)}(\pi_{J(\mathbb{V}^\lambda)}(\lambda))$, a finite-dimensional $\mathfrak{g}_{J(\mathbb{V}^\lambda)}$ -module, are equal. Hence the sets of vertices are the same, so by Theorem 4.3, $W_J(\pi_{J(\mathbb{V}^\lambda)}(\lambda)) = W_{J'}(\pi_{J(\mathbb{V}^\lambda)}(\lambda))$ in $\text{wt } M$. But then the same holds in $\text{wt } \mathbb{V}^\lambda$ via $\varpi_{J(\mathbb{V}^\lambda)}$ (using Lemma 4.5).

Conversely, assume (6); again use Lemma 4.5 and work inside $M = L_{J(\mathbb{V}^\lambda)}(\pi_{J(\mathbb{V}^\lambda)}(\lambda))$ (via $\varpi_{J(\mathbb{V}^\lambda)}$). Theorem 4.3 for $\mathfrak{g}_{J(\mathbb{V}^\lambda)}$ shows that $\text{wt}_J M = \text{wt}_{J'} M$, so $\text{wt}_J \mathbb{V}^\lambda = \text{wt}_{J'} \mathbb{V}^\lambda$ via $\varpi_{J(\mathbb{V}^\lambda)}$. Finally, (7) \implies (1) using Lemma 4.9 (below), and conversely, $X := \text{wt}_J \mathbb{V}^\lambda = \text{wt}_{J'} \mathbb{V}^\lambda$ is stable under both W_J and $W_{J'}$ by Theorem 2.3. Hence so is ρ_X , which shows (7). \square

The previous proof and the proof of Theorem A use the following two preliminary results.

Lemma 4.8. Fix $\lambda \in \mathfrak{h}^*$, $M(\lambda) \twoheadrightarrow \mathbb{V}^\lambda$, and $I_0 \subset I$ such that $\mathbb{V}_{I_0}^\lambda := U(\mathfrak{g}_{I_0})v_\lambda$ is a simple \mathfrak{g}_{I_0} -module. Then the following are equivalent for $J \subset I_0$:

- (1) $\text{wt}_J \mathbb{V}_{I_0}^\lambda = \text{wt}_\emptyset \mathbb{V}_{I_0}^\lambda = \{\lambda\}$.
- (2) $\lambda - \alpha_j \notin \text{wt}_J \mathbb{V}_{I_0}^\lambda \ \forall j \in J$.
- (3) $x_j^- v_\lambda = 0 \ \forall j \in J$.
- (4) $x_j^- v_\lambda \in \ker \mathfrak{n}^+ \ \forall j \in J$.
- (5) $J \subset I \setminus \text{supp}(\lambda)$, i.e., $(\lambda, \alpha_j) = 0 \ \forall j \in J$.

Moreover, if $J \cap \text{supp}(\lambda) \neq J' \cap \text{supp}(\lambda)$ (for $J, J' \subset I_0$), then $\text{wt}_J \mathbb{V}^\lambda \neq \text{wt}_{J'} \mathbb{V}^\lambda$. In particular, the assignment $: J \mapsto \text{wt}_J \mathbb{V}^\lambda$ is one-to-one on the power set of $I_0 \cap \text{supp}(\lambda)$.

A special case is $\mathbb{V}_{I_0}^\lambda = L(\lambda)$ (for any $\lambda \in \mathfrak{h}^*$), when $\mathbb{V}^\lambda = L(\lambda)$ and $I_0 = I$.

Proof. That (1) \implies (2) \implies (3) \implies (4) is clear. Next, given (4), $0 = x_{\alpha_j}^+ x_{\alpha_j}^- v_\lambda = \lambda(h_j) v_\lambda$, whence $\lambda(h_j) = 0$. Thus $(\lambda, \alpha_j) = 0 \ \forall j \in J$, whence $J \subset I \setminus \text{supp}(\lambda)$.

We now show all the contrapositives. Suppose $\lambda > \mu = \lambda - \sum_{j \in J} a_j \alpha_j \in \text{wt}_J \mathbb{V}_{I_0}^\lambda = \text{wt}_J \mathbb{V}^\lambda$ (by Lemma 4.5). By Lemma 4.4, there exists a sequence $\lambda = \mu_0 > \mu_1 > \dots > \mu_N = \mu$ in $\text{wt}_J \mathbb{V}^\lambda$, such that $\mu_j - \mu_{j+1} \in \Delta_J \ \forall j$. Thus, $\mu_1 = \lambda - \alpha_j \in \text{wt } \mathbb{V}^\lambda$ for some $j \in J$, which contradicts (2). In turn, this implies:

$x_{\alpha_j}^- v_\lambda \neq 0$ (notation as in Lemma 4.4), which contradicts (3). If (3) fails, then $x_{\alpha_j}^- v_\lambda$ is not a maximal vector (i.e., not in $\ker \mathfrak{n}^+$), since $\mathbb{V}_{I_0}^\lambda$ is simple. If (4) is false, then by the Serre relations, $0 \neq x_{\alpha_j}^+ x_{\alpha_j}^- v_\lambda = \lambda(h_j) v_\lambda$. Hence $(\lambda, \alpha_j) \neq 0$, i.e., $j \in \text{supp}(\lambda)$. This contradicts (5).

Finally, given $J, J' \subset I_0$ as above, choose $j \in J \cap \text{supp}(\lambda) \setminus J'$. By the above equivalences (in which $J = \{j\}$), $\lambda - \alpha_j \in \text{wt } \mathbb{V}_{I_0}^\lambda$. Hence $\lambda - \alpha_j \in \text{wt}_J \mathbb{V}_{I_0}^\lambda \setminus \text{wt}_{J'} \mathbb{V}_{I_0}^\lambda$, whence $\text{wt}_J \mathbb{V}_{I_0}^\lambda \neq \text{wt}_{J'} \mathbb{V}_{I_0}^\lambda$. By Lemma 4.5, $\text{wt}_J \mathbb{V}^\lambda \neq \text{wt}_{J'} \mathbb{V}^\lambda$ (since $J, J' \subset I_0$). \square

Lemma 4.9. *Suppose either that the setup of Proposition 4.1 holds and $W_{J \cup J'}$ fixes $\rho_{\text{wt}_{J'} \mathbb{V}^\lambda}$; or suppose $J' \subset I$, $J \subset J(\mathbb{V}^\lambda)$, and Δ_J is orthogonal to λ and to $\Delta_{J'}$. Then $\text{wt}_{J'} \mathbb{V}^\lambda = \text{wt}_{J \cup J'} \mathbb{V}^\lambda$.*

Proof. First assume that the setup of Proposition 4.1 holds. Suppose the conclusion fails, i.e.,

$$(4.10) \quad \mu = \lambda - \sum_{j \in J'} a_j \alpha_j - \sum_{j \in J \setminus J'} a_j \alpha_j \in \text{wt}_{J \cup J'} \mathbb{V}^\lambda \setminus \text{wt}_{J'} \mathbb{V}^\lambda.$$

As in the proof of Lemma 4.4, produce a monomial word $0 \neq x_{\alpha_{i_N}}^- \cdots x_{\alpha_{i_1}}^- v_\lambda \in \mathbb{V}_\mu^\lambda$. Then all indices are in $J \cup J'$; choose the smallest k such that $i_k \in J \setminus J'$, and define $\mu_{k-1} := \lambda - \sum_{l=1}^{k-1} \alpha_{i_l} \in \text{wt}_{J'} \mathbb{V}^\lambda$. Now, $(\mu_{k-1}, \alpha_{i_k}) = (\lambda, \alpha_{i_k}) - \sum_{l=1}^{k-1} (\alpha_{i_l}, \alpha_{i_k})$, and each term in the sum is nonpositive since $i_l \in J'$, $i_k \in J \setminus J'$. Since $\alpha_{i_k} \in \Delta_{J \setminus J'} \subset \Delta_{J(\mathbb{V}^\lambda)}$, hence $(\mu_{k-1}, \alpha_{i_k}) \geq 0$.

We first claim that $(\mu_{k-1}, \alpha_{i_k}) > 0$. Suppose not. Then $(\lambda, \alpha_{i_k}) = (\alpha_{i_l}, \alpha_{i_k}) = 0 \ \forall l < k$, whence by the Serre relations, $[x_{\alpha_{i_l}}^-, x_{\alpha_{i_k}}^-] = 0 \ \forall l < k$. Hence by the previous paragraph,

$$(4.11) \quad 0 \neq x_{\alpha_{i_k}}^- \cdots x_{\alpha_{i_1}}^- v_\lambda = x_{\alpha_{i_{k-1}}}^- \cdots x_{\alpha_{i_1}}^- x_{\alpha_{i_k}}^- v_\lambda.$$

In particular, $x_{\alpha_{i_k}}^- v_\lambda \neq 0$. But this contradicts Lemma 4.8 (with $J = I_0 = \{i_k\} \subset J(\mathbb{V}^\lambda)$), since $(\lambda, \alpha_{i_k}) = 0$. This proves the claim. Moreover, as shown above for μ_{k-1} , $(\mu, \alpha_{i_k}) \geq 0 \ \forall \mu \in \text{wt}_{J'} \mathbb{V}^\lambda$. Hence $(\rho_{\text{wt}_{J'} \mathbb{V}^\lambda}, \alpha_{i_k}) > 0$ from the above analysis. But this contradicts the $W_{J \cup J'}$ -invariance of $\rho_{\text{wt}_{J'} \mathbb{V}^\lambda}$, since $\alpha_{i_k} \in \Delta_{J \setminus J'} \subset \Delta_{J \cup J'}$. This shows the first assertion.

The second assertion is shown by essentially repeating the above proof; here is a quick sketch. Suppose again that $\mu \in \text{wt } \mathbb{V}^\lambda$ satisfies (4.10). Produce a monomial word $0 \neq x_{\alpha_{i_N}}^- \cdots x_{\alpha_{i_1}}^- v_\lambda \in \mathbb{V}_\mu^\lambda$. Choose the smallest index k such that $i_k \in J \setminus J'$; then (4.11) holds as well, since $(\alpha_{i_k}, \alpha_{i_l}) = 0$ for all $0 < l < k$ by assumption. Now since $i_k \in J \subset J(\mathbb{V}^\lambda)$ and $(\lambda, \alpha_{i_k}) = 0$, it follows that $x_{\alpha_{i_k}}^- v_\lambda = 0$ in \mathbb{V}^λ , which contradicts (4.11). Thus no weight μ of the form (4.10) exists. \square

4.2. Proof of Theorem A. Having proved Proposition 4.1, we can show our (first) main result.

Proof of Theorem A. First note that the sets of simple roots used in the formulas in Equations 3.2 and 3.3 indeed depend only on $\text{supp}(\lambda)$, J , and $J \setminus J(\mathbb{V}^\lambda) = J_2(\mathbb{V}^\lambda)$. Now let J_{\min}, J_{\max} denote the (first) expressions on the right-hand sides of equations (3.2), (3.3) respectively. We now prove the various implications in the result, and also show the minimality and maximality of these expressions J_{\min}, J_{\max} respectively. The proof is divided into steps for ease of exposition.

(2) \iff (3). We first record the following fact, and use it without reference in the rest of the paper. Given $k > 0$ and $J'_r, J''_s \subset I$,

$$(4.12) \quad \bigcap_{r=1}^k \text{wt}_{J'_r} \mathbb{V}^\lambda \cap \bigcap_s \text{conv}_{\mathbb{R}}(\text{wt}_{J''_s} \mathbb{V}^\lambda) = \text{wt}_{\cap_r J'_r \cap_s J''_s} \mathbb{V}^\lambda.$$

Now clearly (2) \implies (3); the converse follows because $(\text{wt} \mathbb{V}^\lambda) \cap \text{conv}_{\mathbb{R}}(\text{wt}_J \mathbb{V}^\lambda) = \text{wt}_J \mathbb{V}^\lambda$. For the same reason, the assertions in this theorem are also equivalent to the following statement:

(6) *There exist $w, w' \in W_{J(\mathbb{V}^\lambda)}$ such that $w(\text{conv}_{\mathbb{R}} \text{wt}_J \mathbb{V}^\lambda) = w'(\text{conv}_{\mathbb{R}} \text{wt}_{J'} \mathbb{V}^\lambda)$.*

(5) \implies (2). Note by (5) that

$$\text{wt}_{J_{\min} \sqcup (J \setminus J(\mathbb{V}^\lambda))} \mathbb{V}^\lambda \subset \text{wt}_{J'} \mathbb{V}^\lambda \subset \text{wt}_{J_{\max} \sqcup (J \setminus J(\mathbb{V}^\lambda))} \mathbb{V}^\lambda.$$

We now claim that the first and third terms in this chain are equal. Indeed, note by definition of the sets $J_i(\mathbb{V}^\lambda)$ that (with a slight abuse of notation,) $J_5(\mathbb{V}^\lambda)$ is orthogonal to $\{\lambda\} \cup J_2(\mathbb{V}^\lambda) \cup J_3(\mathbb{V}^\lambda) \cup J_4(\mathbb{V}^\lambda)$. It follows from equations (3.2), (3.3) that $\Delta_{J_{\max} \setminus J_{\min}}$ is contained in $\Delta_{J(\mathbb{V}^\lambda)}$ and orthogonal to $\{\lambda\} \sqcup \Delta_{J_{\min} \sqcup J_2(\mathbb{V}^\lambda)}$. Applying the second part of Lemma 4.9 then yields the claim, since

$$(4.13) \quad \text{wt}_{J_{\min} \sqcup (J \setminus J(\mathbb{V}^\lambda))} \mathbb{V}^\lambda = \text{wt}_{J'} \mathbb{V}^\lambda = \text{wt}_{J_{\max} \sqcup (J \setminus J(\mathbb{V}^\lambda))} \mathbb{V}^\lambda.$$

In particular, the previous equality holds for $J' = J$ (since (5) does too), and this shows (2).

(2) \implies (5). First note via Theorem 2.3 that $J \setminus J(\mathbb{V}^\lambda) = \{i \in I : \lambda - \mathbb{Z}_+ \alpha_i \subset \text{wt}_J \mathbb{V}^\lambda\}$. Thus (2) implies that $J \setminus J(\mathbb{V}^\lambda) = J' \setminus J(\mathbb{V}^\lambda)$. Now suppose $C \subset J_3(\mathbb{V}^\lambda) \subset J$ is a connected component of $J \cap J(\mathbb{V}^\lambda)$, such that $\pi_C(\lambda) \neq 0$. Then

$$\text{wt}_{C \cap J'} \mathbb{V}^\lambda = \text{wt}_C \mathbb{V}^\lambda \cap \text{wt}_{J'} \mathbb{V}^\lambda = \text{wt}_C \mathbb{V}^\lambda \cap \text{wt}_J \mathbb{V}^\lambda = \text{wt}_C \mathbb{V}^\lambda,$$

and by [KR, Proposition 5.1], the affine hull of $\text{wt}_C \mathbb{V}^\lambda$ is $\lambda - \mathbb{R} \Delta_C$. Hence the same holds for $\text{wt}_{C \cap J'} \mathbb{V}^\lambda$, whence $C \subset J'$. It follows that $J_3(\mathbb{V}^\lambda) \subset J'$.

Similarly, suppose $C \subset J_4(\mathbb{V}^\lambda)$ is a connected component of $J \cap J(\mathbb{V}^\lambda)$, such that $\pi_C(\lambda) = 0 \neq (\Delta_{J \setminus J(\mathbb{V}^\lambda)}, \Delta_C)$. Assume $(\alpha_{j_2}, \alpha_c) \neq 0$ for some $j_2 \notin J(\mathbb{V}^\lambda)$ and $c \in C$. Since $C \cap (J \setminus J(\mathbb{V}^\lambda))$ is empty, it follows that $(\lambda - \alpha_{j_2}, \alpha_c) > 0$. Now consider the highest weight \mathfrak{g}_C -module $M := U(\mathfrak{g}_C) x_{\alpha_{j_2}}^- v_\lambda$. By integrability, M is finite-dimensional, hence isomorphic to $L_C(\pi_C(\lambda - \alpha_{j_2}))$ as \mathfrak{g}_C -modules. Since C is connected and $\pi_C(\lambda - \alpha_{j_2}) \in P_C^+ \setminus \{0\}$ by the above calculation, once again use [KR, Proposition 5.1] to obtain that the affine hull of $\text{wt} M$ is $\lambda - \mathbb{R} \Delta_C$. Hence the same holds for $\text{wt}_C \mathbb{V}^\lambda \cap \text{wt}_{J'} \mathbb{V}^\lambda = \text{wt}_{C \cap J'} \mathbb{V}^\lambda$, whence $C \subset J'$. It follows that $J_4(\mathbb{V}^\lambda) \subset J'$. Putting together the above analysis shows that $J_{\min} \subset J'$; i.e., the expression in equation (3.2) is indeed minimal as claimed. It also follows that if $\pi_C(\lambda) \neq 0$ or if Δ_C is not orthogonal to α_{j_2} for $j_2 \in J_2(\mathbb{V}^\lambda)$, then $\pi_C(\lambda - \alpha_{j_2}) \neq 0$. Therefore $C \subset J_3(M(\lambda - \alpha_{j_2}, J(\mathbb{V}^\lambda)))$, which implies the second equality in (3.2).

Finally, we claim that $J(\mathbb{V}^\lambda) \setminus J_{\max}$ is disjoint from J' . The claim would imply that $J' \cap J(\mathbb{V}^\lambda) \subset J_{\max}$, which would complete the proof that (2) \implies (5), and also prove that the penultimate expression in equation (3.3) is maximal as asserted.

To show the claim, fix an element

$$\begin{aligned} j \in J(\mathbb{V}^\lambda) \setminus J_{\max} &= J_6(\mathbb{V}^\lambda) \setminus J_{\max} \\ &= (J_6(\mathbb{V}^\lambda) \setminus \{\lambda\}^\perp) \cup ((J_6(\mathbb{V}^\lambda) \cap \{\lambda\}^\perp) \setminus J_2(\mathbb{V}^\lambda)^\perp) \\ &\quad \cup ((J_6(\mathbb{V}^\lambda) \cap \{\lambda\}^\perp) \setminus J_{\min}^\perp). \end{aligned}$$

We show in each of these three cases that $j \notin J'$, which would complete the proof. First if $j \in J_6(\mathbb{V}^\lambda) \setminus \{\lambda\}^\perp$, then $s_j(\lambda) \in \text{wt } \mathbb{V}^\lambda \setminus \text{wt}_J \mathbb{V}^\lambda = \text{wt } \mathbb{V}^\lambda \setminus \text{wt}_{J'} \mathbb{V}^\lambda$. Thus $j \notin J'$. Similarly if $j \in (J_6(\mathbb{V}^\lambda) \cap \{\lambda\}^\perp) \setminus J_2(\mathbb{V}^\lambda)^\perp$, then choose $j' \in J \setminus J(\mathbb{V}^\lambda) = J_2(\mathbb{V}^\lambda)$ such that $(\alpha_j, \alpha_{j'}) \neq 0$. Now $\lambda - \alpha_{j'} \in \text{wt } \mathbb{V}^\lambda$ and $j \in J(\mathbb{V}^\lambda)$, so Theorem 2.3 yields

$$s_j(\lambda - \alpha_{j'}) = \lambda - s_j(\alpha_{j'}) \in \text{wt } \mathbb{V}^\lambda \setminus \text{wt}_J \mathbb{V}^\lambda = \text{wt } \mathbb{V}^\lambda \setminus \text{wt}_{J'} \mathbb{V}^\lambda.$$

Once again, it follows that $j \notin J'$. Finally, suppose $j \in (J_6(\mathbb{V}^\lambda) \cap \{\lambda\}^\perp) \setminus J_{\min}^\perp$. Choose $j_0 \in J_{\min}$ such that $(\alpha_j, \alpha_{j_0}) \neq 0$. By Equation (3.2) proved above, there are now two cases:

- The first possibility is that $j_0 \in J_3(\mathbb{V}^\lambda)$, i.e., j_0 is in a connected component C of $J \cap J(\mathbb{V}^\lambda)$ such that $\pi_C(\lambda) \neq 0$. In this case write $\lambda - w_\circ^C(\lambda) = \sum_{c \in C} n_c \alpha_c$ for $n_c \in \mathbb{Z}_+$. Then $n_c > 0$ for all c by [KR, Proposition 5.1]; in particular, $n_{j_0} > 0$. Now by Theorem 2.3, $s_j(w_\circ^C(\lambda)) \in \text{wt } \mathbb{V}^\lambda \setminus \text{wt}_J \mathbb{V}^\lambda = \text{wt } \mathbb{V}^\lambda \setminus \text{wt}_{J'} \mathbb{V}^\lambda$.
- Otherwise $j_0 \in J_4(\mathbb{V}^\lambda)$, i.e., j_0 is in a connected component C of $J \cap J(\mathbb{V}^\lambda)$ for which $\pi_C(\lambda) = 0$ but $\Delta_C \not\subseteq \Delta_{J_2(\mathbb{V}^\lambda)}$. Suppose $(\Delta_C, \alpha_{j_2}) \neq 0$ for some $j_2 \in J_2(\mathbb{V}^\lambda) = J \setminus J(\mathbb{V}^\lambda)$. Consider the highest weight \mathfrak{g}_C -module $M := U(\mathfrak{g}_C) x_{\alpha_{j_2}}^- v_\lambda$. By integrability, $\dim M < \infty$ and $M \cong L_C(\pi_C(\lambda - \alpha_{j_2}))$ as \mathfrak{g}_C -modules. Since C is connected and $\pi_C(\lambda - \alpha_{j_2}) \in P_C^+ \setminus \{0\}$ by choice of j_2 , it follows via [KR, Proposition 5.1] that the affine hull of $\text{wt } M$ is $\lambda - \mathbb{R}\Delta_C$. Now write the difference of the extremal weights of M as a sum of positive roots; thus, $(\lambda - \alpha_{j_2}) - w_\circ^C(\lambda - \alpha_{j_2}) = \sum_{c \in C} n_c \alpha_c$, with $n_c > 0$ for all $c \in C$. In particular, $n_{j_0} > 0$. Recall that $j \in J_6(\mathbb{V}^\lambda) \subset J(\mathbb{V}^\lambda)$, so $\mu := s_j(w_\circ^C(\lambda - \alpha_{j_2}))$ lies in $W_{J(\mathbb{V}^\lambda)}(\lambda - \Delta_{J_2}) \subset \text{wt } \mathbb{V}^\lambda$ by Theorem 2.3. On the other hand, since $(\alpha_j, \alpha_{j_0}) \neq 0$, it follows that $\lambda - \mu = \sum_{i \in J(\mathbb{V}^\lambda)} n_i \alpha_i$ with $n_j > 0$ for $j \in J_6(\mathbb{V}^\lambda) = J(\mathbb{V}^\lambda) \setminus J$. Therefore $\mu \in \text{wt } \mathbb{V}^\lambda \setminus \text{wt}_J \mathbb{V}^\lambda = \text{wt } \mathbb{V}^\lambda \setminus \text{wt}_{J'} \mathbb{V}^\lambda$.

In either case the above analysis shows that $j \notin J'$. This yields $J' \cap J(\mathbb{V}^\lambda) \subset J_{\max}$, proving that (2) \implies (5). The equivalence (2) \iff (5) also shows that the expressions in Equation (3.2) and the first expression in Equation (3.3) are indeed the desired, extremal subsets of weights. It is now easily verified that the last two expressions in Equation (3.3) are equal, since $J_5(\mathbb{V}^\lambda) \subset J(\mathbb{V}^\lambda) \cap \{\lambda\}^\perp \cap J_{\min}^\perp \cap J_2(\mathbb{V}^\lambda)^\perp$.

(2) \iff (4). Clearly (4) \implies (2). Conversely, we first *claim* that $U(\mathfrak{g}_J)v_\lambda = U(\mathfrak{g}_{J_{\min} \sqcup (J \setminus J(\mathbb{V}^\lambda))})v_\lambda$ for all $J \subset I$. Note that the claim, together with (2) \iff (5), immediately implies (4).

The claim is proved by showing that each side is contained in the other. One inclusion is obvious; conversely, $U(\mathfrak{g}_J)v_\lambda$ is spanned by the set \mathcal{F}_J of words in the alphabet $\{f_j : j \in J\}$, applied to v_λ . Consider a subset $\mathcal{B} \subset \mathcal{F}_J$ that corresponds

to a weight basis of $U(\mathfrak{g}_J)v_\lambda$. Then using (5),

$$\text{wt } b \in -\lambda + \text{wt}_J \mathbb{V}^\lambda = -\lambda + \text{wt}_{J_{\min} \sqcup (J \setminus J(\mathbb{V}^\lambda))} \mathbb{V}^\lambda \subset -\mathbb{Z}_+ \Delta_{J_{\min} \sqcup (J \setminus J(\mathbb{V}^\lambda))}, \quad \forall b \in \mathcal{B}.$$

Consequently, $\mathcal{B} \subset \mathcal{F}_{J_{\min} \sqcup (J \setminus J(\mathbb{V}^\lambda))}$. This shows that $U(\mathfrak{g}_J)v_\lambda$ is contained in $U(\mathfrak{g}_{J_{\min} \sqcup (J \setminus J(\mathbb{V}^\lambda))})v_\lambda$. This proves the claim, and hence that (2) \implies (4).

(1) \iff (2). Clearly (2) \implies (1). Conversely, suppose (1) holds and $j \in J' \setminus J(\mathbb{V}^\lambda)$. Then by Theorem 2.3 and [Kh, Proposition 4.4],

$$\begin{aligned} w^{-1}w'(\lambda) - \mathbb{Z}_+(w^{-1}w'\alpha_j) &= w^{-1}w'(\lambda - \mathbb{Z}_+\alpha_j) \subset w^{-1}w'(\text{wt}_{J'} \mathbb{V}^\lambda) \\ &= \text{wt}_J \mathbb{V}^\lambda \subset \text{wt}_J M(\lambda, J(\mathbb{V}^\lambda)). \end{aligned}$$

By [KR, Proposition 2.3], this implies that $w^{-1}w'(\alpha_j) \in \mathbb{Z}_+ \Delta_J \cap \mathbb{Z}_+(\Phi^+ \setminus \Phi_{J(\mathbb{V}^\lambda)}^+)$. Therefore,

$$w^{-1}w'(\alpha_j) \in \Phi \cap (\mathbb{Z}_+ \Delta_J) \cap (\Phi^+ \setminus \Phi_{J(\mathbb{V}^\lambda)}^+) = \Phi_J^+ \setminus \Phi_{J \cap J(\mathbb{V}^\lambda)}^+ \subset \Phi_{J \cup J(\mathbb{V}^\lambda)}.$$

This implies that $\alpha_j \in W_{J(\mathbb{V}^\lambda)}(\Phi_{J \cup J(\mathbb{V}^\lambda)}) = \Phi_{J \cup J(\mathbb{V}^\lambda)}$ for all $j \in J' \setminus J(\mathbb{V}^\lambda)$. In particular, $J' \setminus J(\mathbb{V}^\lambda) \subset J \setminus J(\mathbb{V}^\lambda)$, and by symmetry, the reverse inclusion holds as well.

The meat of this implication is in the claim that $J_{\min} \subset J' \cap J(\mathbb{V}^\lambda)$. To show the claim we first describe our approach, in order to clarify the subsequent detailed exposition. Choose finite, distinguished $W_{J(\mathbb{V}^\lambda)}$ -stable subsets $T^\mu \subset \text{wt } \mathbb{V}^\lambda$ (for certain weights $\mu \in Q^+$) and define $T_J^\mu := T^\mu \cap \text{wt}_J \mathbb{V}^\lambda$ for $J \subset I$. Then $w(T_J^\mu) = w'(T_{J'}^\mu)$, so that $w^{-1}w'(\pi_{J(\mathbb{V}^\lambda)}(\rho_{T_{J'}^\mu})) = \pi_{J(\mathbb{V}^\lambda)}(\rho_{T_J^\mu})$ (this is only true after transporting the situation via $\varpi_{J(\mathbb{V}^\lambda)}$ to $L_{J(\mathbb{V}^\lambda)}(\pi_{J(\mathbb{V}^\lambda)}(\lambda - \mu))$). But $\pi_{J(\mathbb{V}^\lambda)}(\rho_{T_{J'}^\mu})$ and $\pi_{J(\mathbb{V}^\lambda)}(\rho_{T_J^\mu})$ are both in $P_{J(\mathbb{V}^\lambda)}^+$, so they are equal, whence their maximizer sets in $T_J^\mu, T_{J'}^\mu$ are equal. This will yield the result by Proposition 4.6 by studying specific sets T^μ for various μ .

We now explain the details in the preceding paragraph. Define

$$(4.14) \quad \begin{aligned} \mathbb{T}(\mathbb{V}^\lambda) &:= \{\mu \in Q_{I \setminus J(\mathbb{V}^\lambda)}^+ : \lambda - \mu \in \text{wt } \mathbb{V}^\lambda\}, \\ \forall \mu \in \mathbb{T}(\mathbb{V}^\lambda), \quad T^\mu &:= \text{wt } L_{J(\mathbb{V}^\lambda)}(\lambda - \mu) \subset \text{wt } \mathbb{V}^\lambda. \end{aligned}$$

There is a slight abuse of notation here; note that T^μ is distinct from $\text{wt } L_{J(\mathbb{V}^\lambda)}(\pi_{J(\mathbb{V}^\lambda)}(\lambda - \mu))$, which is contained in $\pi_{J(\mathbb{V}^\lambda)}(\lambda) - \mathbb{R} \Delta_{J(\mathbb{V}^\lambda)}$. Moreover, T^μ is a finite, $W_{J(\mathbb{V}^\lambda)}$ -stable subset of $\text{wt } \mathbb{V}^\lambda$ for all $\mu \in \mathbb{T}(\mathbb{V}^\lambda)$.

Given $J \subset I$, define $T_J^\mu := T^\mu \cap \text{wt}_J \mathbb{V}^\lambda$. Now suppose (1) holds, i.e., $w^{-1}w'(\text{wt}_{J'} \mathbb{V}^\lambda) = \text{wt}_J \mathbb{V}^\lambda$. Intersecting both sides with T^μ yields:

$$T_J^\mu = T^\mu \cap w^{-1}w'(\text{wt}_{J'} \mathbb{V}^\lambda) = w^{-1}w'(T_{J'}^\mu),$$

whence $\rho_{T_J^\mu} = w^{-1}w'(\rho_{T_{J'}^\mu})$. Now applying Lemma 4.5,

$$w^{-1}w'(\varpi_{J(\mathbb{V}^\lambda)}(\rho_{T_{J'}^\mu})) = \varpi_{J(\mathbb{V}^\lambda)}(w^{-1}w'(\rho_{T_{J'}^\mu})) = \varpi_{J(\mathbb{V}^\lambda)}(\rho_{T_J^\mu}).$$

Note that $\varpi_{J(\mathbb{V}^\lambda)}(\rho_{T_J^\mu}) = \rho_{\text{wt}_{J \cap J(\mathbb{V}^\lambda)} M^\mu}$, where $M^\mu := L_{J(\mathbb{V}^\lambda)}(\pi_{J(\mathbb{V}^\lambda)}(\lambda - \mu))$ is a finite-dimensional $\mathfrak{g}_{J(\mathbb{V}^\lambda)}$ -module because $\mu \in I \setminus J(\mathbb{V}^\lambda)$. Similarly for J' in place of J . Hence $w^{-1}w'(\rho_{\text{wt}_{J' \cap J(\mathbb{V}^\lambda)} M^\mu}) = \rho_{\text{wt}_{J \cap J(\mathbb{V}^\lambda)} M^\mu}$. Apply Proposition 4.6 over $\mathfrak{g}_{J(\mathbb{V}^\lambda)}$; then $\rho_{\text{wt}_{J \cap J(\mathbb{V}^\lambda)} M^\mu}, \rho_{\text{wt}_{J' \cap J(\mathbb{V}^\lambda)} M^\mu} \in P_{J(\mathbb{V}^\lambda)}^+$. Since every $W_{J(\mathbb{V}^\lambda)}$ -orbit contains at most one dominant element, $\rho_{\text{wt}_{J \cap J(\mathbb{V}^\lambda)} M^\mu} = \rho_{\text{wt}_{J' \cap J(\mathbb{V}^\lambda)} M^\mu}$. Therefore

by Proposition 4.6 over $\mathfrak{g}_{J(\mathbb{V}^\lambda)}$, the corresponding maximizer subsets in $\text{wt } M^\mu$ are equal:

$$\begin{aligned} \text{wt}_{J \cap J(\mathbb{V}^\lambda)} M^\mu &= (\text{wt } M^\mu)(\rho_{\text{wt}_{J \cap J(\mathbb{V}^\lambda)} M^\mu}) \\ &= (\text{wt } M^\mu)(\rho_{\text{wt}_{J' \cap J(\mathbb{V}^\lambda)} M^\mu}) = \text{wt}_{J' \cap J(\mathbb{V}^\lambda)} M^\mu. \end{aligned}$$

Thus the problem is now reduced to a finite-dimensional situation over the semisimple Lie algebra $\mathfrak{g}_{J(\mathbb{V}^\lambda)}$. Introduce the following notation for convenience:

$$(4.15) \quad \tilde{J} := J \cap J(\mathbb{V}^\lambda), \quad \tilde{J}' := J' \cap J(\mathbb{V}^\lambda), \quad \eta_\mu := \pi_{J(\mathbb{V}^\lambda)}(\lambda - \mu).$$

Using this notation, the above analysis in the present step shows that

$$(4.16) \quad w(\text{wt}_J \mathbb{V}^\lambda) = w'(\text{wt}_{J'} \mathbb{V}^\lambda) \implies \forall \mu \in \mathbb{T}(\mathbb{V}^\lambda), \text{wt}_{\tilde{J}} L_{J(\mathbb{V}^\lambda)}(\eta_\mu) = \text{wt}_{\tilde{J}'} L_{J(\mathbb{V}^\lambda)}(\eta_\mu).$$

Now apply the equivalence (2) \iff (5) of this theorem to equation (4.16). This yields:

$$(4.17) \quad \tilde{J}_3(L_{J(\mathbb{V}^\lambda)}(\eta_\mu)) = \tilde{J}_{\min} = \tilde{J}'_{\min} = \tilde{J}'_3(L_{J(\mathbb{V}^\lambda)}(\eta_\mu)) \subset \tilde{J}' = J' \cap J(\mathbb{V}^\lambda).$$

The final step in proving the claim that $J_{\min} \subset J' \cap J(\mathbb{V}^\lambda)$, is to study equation (4.17) for various special values of μ , namely, $\mu \in \{0\} \sqcup \Delta_{J_2(\mathbb{V}^\lambda)}$. If $\mu = 0$, then $\tilde{J}_3(L_{J(\mathbb{V}^\lambda)}(\eta_0)) = (J \cap J(\mathbb{V}^\lambda))_{\min} = J_3(\mathbb{V}^\lambda)$, so $J_3(\mathbb{V}^\lambda) \subset J' \cap J(\mathbb{V}^\lambda)$. Next, if $\mu = \alpha_j$ for $j \in J_2(\mathbb{V}^\lambda)$, then by the above analysis, $\tilde{J}_3(L_{J(\mathbb{V}^\lambda)}(\eta_j))$ is the union of the connected components C in the Dynkin diagram of $J \cap J(\mathbb{V}^\lambda)$, which satisfy: $\pi_C(\lambda - \alpha_j) \neq 0$. It follows from the definitions that $J_3(\mathbb{V}^\lambda) \sqcup J_4(\mathbb{V}^\lambda) \subset J' \cap J(\mathbb{V}^\lambda)$. By equation (3.2), it follows that $J_{\min} \subset J' \cap J(\mathbb{V}^\lambda)$, which proves the above claim.

The last step is to note that $J_{\min} \sqcup (J \setminus J(\mathbb{V}^\lambda)) \subset J'$ from the above analysis, so by applying equation (4.13), $\text{wt}_J \mathbb{V}^\lambda = \text{wt}_{J_{\min} \sqcup (J \setminus J(\mathbb{V}^\lambda))} \mathbb{V}^\lambda \subset \text{wt}_{J'} \mathbb{V}^\lambda$. The reverse inclusion is proved by symmetry. Therefore (1) \implies (2) holds and the proof is complete. \square

Concluding remarks: negative results. We conclude this section by discussing a couple of related results that are negative. Given Theorem A, it is natural to ask if the condition $\text{wt}_J \mathbb{V}^\lambda = \text{wt}_{J'} \mathbb{V}^\lambda$ is equivalent to the following “simpler” conditions:

$$(4.18) \quad J \setminus J(\mathbb{V}^\lambda) = J' \setminus J(\mathbb{V}^\lambda), \quad \text{wt}_{J \cap J(\mathbb{V}^\lambda)} \mathbb{V}^\lambda = \text{wt}_{J' \cap J(\mathbb{V}^\lambda)} \mathbb{V}^\lambda.$$

The answer to this question is: not always. Indeed $\text{wt}_J \mathbb{V}^\lambda = \text{wt}_{J'} \mathbb{V}^\lambda$ implies (4.18); however, the converse is not true because the sets $J_4(\mathbb{V}^\lambda)$ and $J'_4(\mathbb{V}^\lambda)$ may not coincide. For a concrete example, let $\mathfrak{g} = \mathfrak{sl}_3$ and consider $\lambda = c\omega_2$ for $c \in \mathbb{C}$, and $\mathbb{V}^\lambda = M(\lambda, \{1\}) = U(\mathfrak{g})/U(\mathfrak{g})(\ker \lambda + \mathfrak{n}^+ + \mathbb{C}x_1^-)$. Thus $I = \{1, 2\}$ and $J(\mathbb{V}^\lambda) = \{1\}$. Now it is easily verified that $J = \{1, 2\}$ and $J' = \{2\}$ satisfy equation (4.18), since $\text{wt}_{\{1\}} \mathbb{V}^\lambda = \{\lambda\} = \text{wt}_\emptyset \mathbb{V}^\lambda$. On the other hand, $s_1(\lambda - \alpha_2) = \lambda - \alpha_1 - \alpha_2$, so $\text{wt}_{J'} \mathbb{V}^\lambda = \lambda - \mathbb{Z}_+ \alpha_2 \subsetneq \text{wt}_J \mathbb{V}^\lambda$.

A related observation is that an approach based on equation (4.18) leads only up to $J_3(\mathbb{V}^\lambda)$, while $J_{\min} = J_3(\mathbb{V}^\lambda) \sqcup J_4(\mathbb{V}^\lambda)$. However this is not an obstruction if one recalls that by equation (3.2), J_{\min} can be expressed only using “ J_3 -type” sets for various highest weight modules.

A second question arises upon observing that if $\text{wt}_{J'} \mathbb{V}^\lambda = \text{wt}_{J \cup J'} \mathbb{V}^\lambda$, then obviously $\text{wt}_J \mathbb{V}^\lambda \subset \text{wt}_{J'} \mathbb{V}^\lambda$. Given Theorem A, it is natural to ask if the converse

always holds as well. It turns out that this is not the case; for example, suppose $\mathfrak{g} = \mathfrak{sl}_3$, $\lambda = (c+1)\omega_2 \in P^+ \setminus \{0\}$ with $c \in \mathbb{Z}_+$, and $\mathbb{V}^\lambda = L(\lambda)$ is simple. Then,

$$\text{wt}_{\{1\}} L(\lambda) = \{\lambda\} \subsetneq \text{wt}_{\{2\}} L(\lambda) \subsetneq \text{wt}_{\{1,2\}} L(\lambda) = \text{wt } L(\lambda).$$

5. FACES OF HIGHEST WEIGHT MODULES: COMBINATORIAL RESULTS

In this section we apply Theorem A in order to study highest weight modules in greater detail. The goal of this section is to prove Theorem B. We begin by explaining Remark 3.7, which discussed how the notion of a Weyl polytope was extended in [Kh] to apply to general highest weight modules.

Proposition 5.1. *Suppose $(\lambda, \mathbb{V}^\lambda)$ satisfy any of the four assumptions in Theorem 2.5: (a) $\lambda(h_i) \neq 0 \ \forall i \in I$ and \mathbb{V}^λ is arbitrary; (b) $|J_\lambda \setminus J(\mathbb{V}^\lambda)| \leq 1$ (e.g., if \mathbb{V}^λ is simple for any $\lambda \in \mathfrak{h}^*$); (c) $\mathbb{V}^\lambda = M(\lambda, J')$ for some $J' \subset J_\lambda$; or (d) \mathbb{V}^λ is pure (in the sense of [Fe]).*

Then equation (3.8) holds: $\text{conv}_{\mathbb{R}} \text{wt } \mathbb{V}^\lambda = \text{conv}_{\mathbb{R}} \text{wt } M(\lambda, J(\mathbb{V}^\lambda))$. In turn, equation (3.8) implies all of the conclusions in Theorem 2.5.

Proof. It is not hard to show that both parts of this result follow from the proofs of [Kh, Theorems B and C]. In fact, the condition (3.8) implies all of the conclusions of [Kh, Theorems B and C]. \square

In order to prove Theorem B, two additional preliminary results are required. The first result involves the barycenter of a finite standard parabolic subset of weights of $\text{wt } \mathbb{V}^\lambda$.

Lemma 5.2. *If $\lambda \in \mathfrak{h}^*$, $M(\lambda) \twoheadrightarrow \mathbb{V}^\lambda$, and $J \subset I$, then $\text{avg}(\text{wt}_{J \cap J(\mathbb{V}^\lambda)} \mathbb{V}^\lambda) = \text{avg}(W_{J \cap J(\mathbb{V}^\lambda)}(\lambda))$. In other words, the barycenter of the set $\text{wt}_{J \cap J(\mathbb{V}^\lambda)} \mathbb{V}^\lambda$ coincides with that of the vertices of its convex hull; moreover, this vector lies in $\mathbb{Q}_+ \Omega_{J_\lambda \setminus (J \cap J(\mathbb{V}^\lambda))_{\max}} \times \mathbb{C} \Omega_{I \setminus J_\lambda}$.*

Note that this result specializes to [CM, Theorem 1.2(3)] when \mathfrak{g} is simple and \mathbb{V}^λ is the adjoint representation $\mathfrak{g} = L(\theta)$ (via the dictionary mentioned in Section 8). The result also extends Proposition 4.6(1). Further note that $(J \cap J(\mathbb{V}^\lambda))_{\max}$ can be computed using Theorem A.

Proof. Consider the $\mathfrak{g}_{J(\mathbb{V}^\lambda)}$ -submodule $U(\mathfrak{g}_{J(\mathbb{V}^\lambda)})v_\lambda \cong L_{J(\mathbb{V}^\lambda)}(\lambda)$ of \mathbb{V}^λ . Use Lemma 4.5 and [KR, Proposition 5.2] to obtain that

$$(5.3) \quad \text{avg}(\text{wt}_{J \cap J(\mathbb{V}^\lambda)} \mathbb{V}^\lambda) - (\lambda - \pi_{J(\mathbb{V}^\lambda)}(\lambda)) = \text{avg}(W_{J \cap J(\mathbb{V}^\lambda)}(\lambda)) - (\lambda - \pi_{J(\mathbb{V}^\lambda)}(\lambda)).$$

This proves the first equality. The last assertion follows from Proposition 4.6 and Theorem A, with J replaced by $J \cap J(\mathbb{V}^\lambda)$. \square

The second result proves some of the assertions in Theorem B, including a stabilizer subgroup computation in the finite-dimensional setting.

Proposition 5.4. *Suppose $\lambda \in \mathfrak{h}^*$, $M(\lambda) \twoheadrightarrow \mathbb{V}^\lambda$, and $J \subset I$.*

- (1) *In the Weyl group W , $W_{J_{\max}} = W_{J_{\min}} \times W_{J_{\max} \setminus J_{\min}}$. Here $W_{J_{\max} \setminus J_{\min}}$ fixes the face $\text{conv}_{\mathbb{R}} \text{wt}_J \mathbb{V}^\lambda$ pointwise, while no element of $W_{J_{\max}} \setminus W_{J_{\max} \setminus J_{\min}}$ does so.*
- (2) *Let $J \subset J(\mathbb{V}^\lambda)$. Then the stabilizer subgroups in $W_{J(\mathbb{V}^\lambda)}$ of $\text{conv}_{\mathbb{R}}(\text{wt}_J \mathbb{V}^\lambda)$ and of (the average of) $\text{wt}_J \mathbb{V}^\lambda$ agree, and equal $W_{J_{\max}}$.*

Proof. (1) It follows from equations (3.2) and (3.3) and the definitions that $\Delta_{J_{\max} \setminus J_{\min}}$ is orthogonal to $\{\lambda\} \sqcup \Delta_{J_{\min} \sqcup J_2(\mathbb{V}^\lambda)}$. Therefore by Theorem A, $W_{J_{\max} \setminus J_{\min}}$ fixes $\text{wt}_J \mathbb{V}^\lambda$ (and hence its convex hull) pointwise. It also follows that $W_{J_{\max} \setminus J_{\min}}$ commutes with $W_{J_{\min}}$ in W .

It remains to prove no element of $W_{J_{\max}} \setminus W_{J_{\max} \setminus J_{\min}}$ fixes all of $\text{conv}_{\mathbb{R}} \text{wt}_J \mathbb{V}^\lambda$. Indeed, suppose $w \in W_{J_{\max}}$ fixes $\text{conv}_{\mathbb{R}} \text{wt}_J \mathbb{V}^\lambda$ pointwise. Write $w = w_1 w_2$, where $w_1 \in W_{J_{\min}}$, $w_2 \in W_{J_{\max} \setminus J_{\min}}$. Then w fixes $\text{conv}_{\mathbb{R}} \text{wt}_J \mathbb{V}^\lambda$ pointwise if and only if w_1 does so. We claim this happens if and only if $w_1 = 1$. In fact, we show the stronger statement that no nontrivial $w \in W_{J_{\min}}$ fixes $\text{wt}_J \mathbb{V}^\lambda$.

To show this statement, first note that any $w \in W_{J_{\min}}$ which fixes $\text{wt}_J \mathbb{V}^\lambda$ must fix λ and $\lambda - \alpha_{j_2}$ for all $j_2 \in J_2(\mathbb{V}^\lambda)$, so it fixes $\Delta_{J \setminus J(\mathbb{V}^\lambda)}$. Next, the minimality of $(J \cap J(\mathbb{V}^\lambda))_{\min} = J_3(\mathbb{V}^\lambda)$ implies shows that for all $j_3 \in J_3(\mathbb{V}^\lambda)$, there exists a weight $\mu \in \text{wt}_{J_3(\mathbb{V}^\lambda)} \mathbb{V}^\lambda \subset \text{wt}_J \mathbb{V}^\lambda$ such that $\lambda - \mu = \sum_{j \in J_3(\mathbb{V}^\lambda)} c_j \alpha_j$ with $c_{j_3} > 0$. Using Lemma 4.4, it follows that w fixes α_{j_3} for all $j_3 \in J_3(\mathbb{V}^\lambda)$. Next, if $j_4 \in J_4(\mathbb{V}^\lambda)$ then let C be the connected component of the Dynkin diagram of $J \cap J(\mathbb{V}^\lambda)$ such that $j_4 \in C \subset J_4(\mathbb{V}^\lambda)$. Choose $j_2 \in J_2(\mathbb{V}^\lambda)$ such that $(\alpha_{j_2}, \alpha_{j_4}) \neq 0$; then $\pi_C(\lambda - \alpha_{j_2}) \neq 0$. Now recall from the proof of (1) \implies (2) in Theorem A that $\text{wt}_{L_{J(\mathbb{V}^\lambda)}}(\lambda - \alpha_{j_2}) \subset \text{wt}_J \mathbb{V}^\lambda$. Since w fixes λ as well as $\Delta_{J_2(\mathbb{V}^\lambda) \sqcup J_3(\mathbb{V}^\lambda)}$, an argument similar to that for $J_3(\mathbb{V}^\lambda)$ above shows that w also fixes α_{j_4} , and hence all of $\Delta_{J_4(\mathbb{V}^\lambda)}$. It follows by equation (3.2) that $w \in W_{J_{\min}}$ fixes $\Delta_{J_{\min}}$, and hence sends no positive root in $\Phi_{J_{\min}}$ to Φ^- . Therefore w has length zero in $W_{J_{\min}}$, i.e. $w = 1$ as claimed.

(2) Define the sets $S_j \subset \mathfrak{h}^*$ for $1 \leq j \leq 5$ as follows:

$$\begin{aligned} S_1 &:= \text{wt}_J \mathbb{V}^\lambda, & S_2 &:= \text{conv}_{\mathbb{R}} \text{wt}_{J_{\max}} \mathbb{V}^\lambda, & S_3 &:= W_{J_{\max}}(\lambda), \\ S_4 &:= \{\text{avg}(S_1)\}, & S_5 &:= \{\pi_{J(\mathbb{V}^\lambda)}(\text{avg}(S_1))\}. \end{aligned}$$

Now define $W_j := \text{stab}_{W_{J(\mathbb{V}^\lambda)}} S_j$ to be the respective stabilizer subgroup in $W_{J(\mathbb{V}^\lambda)}$ of S_j , for $1 \leq j \leq 5$. We then claim that $W_{J_{\max}} \subset W_1 \subset \cdots \subset W_5 \subset W_{J_{\max}}$, which shows that these subgroups are all equal and proves this part.

To prove the claim, note by Theorem A that $S_1 = \text{wt}_{J_{\max}} \mathbb{V}^\lambda$ is stable under $W_{J_{\max}}$, so that $W_{J_{\max}} \subset W_1$. Next, it is clear that $W_1 \subset W_2$, and Proposition 4.1 (with J replaced by J_{\max}) shows that $W_2 \subset W_3$. Moreover, equation (5.3) shows that $\text{avg} \text{wt}_J \mathbb{V}^\lambda = \text{avg} \text{wt}_{J_{\max}} \mathbb{V}^\lambda = \text{avg} S_3$. Hence $W_3 \subset W_4$. Next, write $\text{avg}(S_1) = \pi_{J(\mathbb{V}^\lambda)}(\text{avg}(S_1)) + \pi_{I \setminus J(\mathbb{V}^\lambda)}(\text{avg}(S_1))$. Note that $W_4 \subset W_{J(\mathbb{V}^\lambda)}$ fixes the first and third vectors in this equation. Hence $W_4 \subset W_5$.

It remains to show $W_5 \subset W_{J_{\max}}$. Note by Lemma 4.5, one can reduce the problem to the case where I, W , and \mathfrak{g} are equal to $J(\mathbb{V}^\lambda), W_{J(\mathbb{V}^\lambda)}$, and $\mathfrak{g}_{J(\mathbb{V}^\lambda)}$ respectively. Now Lemma 5.2 implies that $|S_1| \text{avg}(S_1) \in \mathbb{Z}_+ \Omega_{I \setminus J_{\max}}$ lies in the dominant Weyl chamber. It follows by assertion (I) in [Bou, Chapter V.3.3] and Theorem 2 in [Bou, Chapter VI.1.5] that W_5 is generated by the simple reflections it contains. Denote the indices corresponding to these simple reflections by J_0 ; thus, $J_0 := \{i \in I : (\text{avg}(S_1), \alpha_i) = 0\}$. Now note by [KR, Proposition 5.2] as well as Proposition 4.6 that $(\text{avg}(S_1), -)$ is maximized precisely at $\text{wt}_{J_{\max}} \mathbb{V}^\lambda$. Since J_{\max} is maximal in the sense of Theorem A, it follows that $J_0 \subset J_{\max}$, whence $W_5 = W_{J_0} \subset W_{J_{\max}}$. \square

We now use Theorem A as well as the above analysis in the present section, to show another of the main results in this paper.

Proof of Theorem B. (1) Note that the affine hull of $\text{conv}_{\mathbb{R}} \text{wt}_J \mathbb{V}^\lambda$ equals the λ -translate of the real span of the set $S_{J,\lambda} := \lambda - \text{wt}_J \mathbb{V}^\lambda = \lambda - \text{wt}_{J_{\min} \sqcup (J \setminus J(\mathbb{V}^\lambda))} \mathbb{V}^\lambda$ by Theorem A. This implies that $\text{span}_{\mathbb{R}}(S_{J,\lambda}) \subset \mathbb{R}\Delta_{J_{\min} \sqcup (J \setminus J(\mathbb{V}^\lambda))}$. Now note that $\mathbb{Z}_+ \alpha_{j_2} \in S_{J,\lambda}$ for all $j_2 \in J \setminus J(\mathbb{V}^\lambda)$. Moreover, the minimality of $(J \cap J(\mathbb{V}^\lambda))_{\min} = J_3(\mathbb{V}^\lambda)$ implies that for all $j_3 \in J_3(\mathbb{V}^\lambda)$, there exists a weight $\mu_{j_3} \in S_{J,\lambda}$ such that $\mu_{j_3} = \sum_{j \in J_{\min}} c_j \alpha_j$ with $c_{j_3} > 0$. Using Lemma 4.4, it follows that $\alpha_{j_3} \in \text{span}_{\mathbb{R}}(S_{J,\lambda})$ for each $j_3 \in J_{\min}$.

Finally, suppose $j_4 \in C \subset J_4(\mathbb{V}^\lambda)$, where C is a connected component of the Dynkin diagram of $J \cap J(\mathbb{V}^\lambda)$. Then there exists $j_2 \in J_2(\mathbb{V}^\lambda)$ such that $(\alpha_{j_2}, \alpha_{j_4}) \neq 0$. Now recall from the proof of (1) \implies (2) in Theorem A that $\text{wt}_{L_{J(\mathbb{V}^\lambda)}}(\lambda - \alpha_{j_2}) \subset \text{wt } \mathbb{V}^\lambda$ by the $W_{J(\mathbb{V}^\lambda)}$ -integrability of \mathbb{V}^λ . Since $\pi_C(\lambda - \alpha_{j_2}) \neq 0$, it follows similar to the above reasoning for $J_3(\mathbb{V}^\lambda)$ that there exists $\mu_{j_4} \in S_{J,\lambda}$ of the form $\mu_{j_4} = \sum_{j \in J_{\min}} c_j \alpha_j$ with $c_{j_4} > 0$. Therefore $\Delta_{J_4(\mathbb{V}^\lambda)} \subset S_{J,\lambda}$, and hence, $\text{span}_{\mathbb{R}}(S_{J,\lambda}) = \mathbb{R}\Delta_{J_{\min} \sqcup (J \setminus J(\mathbb{V}^\lambda))}$. Taking dimensions of both sides completes the proof of this part.

(2) This part is the meat of the proof. We first claim that the stabilizers in $W_{J(\mathbb{V}^\lambda)}$ of $\text{wt}_J \mathbb{V}^\lambda$ and $\text{conv}_{\mathbb{R}} \text{wt}_J \mathbb{V}^\lambda$ agree. Clearly if $w \in W_{J(\mathbb{V}^\lambda)}$ stabilizes $\text{wt}_J \mathbb{V}^\lambda$ then it stabilizes its convex hull. Conversely, if $w \in W_{J(\mathbb{V}^\lambda)}$ stabilizes $\text{conv}_{\mathbb{R}} \text{wt}_J \mathbb{V}^\lambda$, then it stabilizes $(\text{wt } \mathbb{V}^\lambda) \cap \text{conv}_{\mathbb{R}} \text{wt}_J \mathbb{V}^\lambda = \text{wt}_J \mathbb{V}^\lambda$, which shows that the two stabilizer subgroups in $W_{J(\mathbb{V}^\lambda)}$ are equal.

Denote this common stabilizer subgroup in $W_{J(\mathbb{V}^\lambda)}$ by W' . We now claim that $W' = W_{J_{\max}}$. One inclusion is clear: $W_{J_{\max} \setminus J_{\min}}$ fixes $\text{wt}_J \mathbb{V}^\lambda$; moreover, $W_{J_{\min}}$ preserves $\text{wt}_J \mathbb{V}^\lambda = \text{wt } \mathbb{V}^\lambda \cap \lambda - \mathbb{Z}_+ \Delta_J$, since $J_{\min} \subset J$. Therefore $W_{J_{\max}}$ preserves $\text{wt}_J \mathbb{V}^\lambda$. To show the converse inclusion, suppose $w \in W'$ preserves $\text{wt}_J \mathbb{V}^\lambda$. Then w preserves each $W_{J(\mathbb{V}^\lambda)}$ -stable subset of $\text{wt}_J \mathbb{V}^\lambda$. Now recall the notation in equation (4.14); thus w preserves the sets $T^\mu \cap \text{wt}_J \mathbb{V}^\lambda$ for $\mu \in \mathbb{T}(\mathbb{V}^\lambda)$. Therefore one can use Lemma 4.5 to transfer the problem to $\text{wt}_{J(\mathbb{V}^\lambda)} L_{J(\mathbb{V}^\lambda)}(\pi_{J(\mathbb{V}^\lambda)}(\lambda - \mu))$. Then w preserves the set

$$\varpi_{J(\mathbb{V}^\lambda)}(T_J^\mu) = \text{wt}_{J \cap J(\mathbb{V}^\lambda)} L_{J(\mathbb{V}^\lambda)}(\pi_{J(\mathbb{V}^\lambda)}(\lambda - \mu)) = \text{wt}_{\tilde{J}_{\max}} L_{J(\mathbb{V}^\lambda)}(\eta_\mu),$$

where \tilde{J} and η_μ were defined in equation (4.15). Now denote the stabilizer in $W_{J(\mathbb{V}^\lambda)}$ of $\varpi_{J(\mathbb{V}^\lambda)}(T^\mu \cap \text{wt}_J \mathbb{V}^\lambda)$ by W''_μ ; then $W_{J_{\max}} \subset W' \subset W''_\mu$. Thus it suffices to show that

$$(5.5) \quad \bigcap_{\mu \in \mathbb{T}(\mathbb{V}^\lambda)} W''_\mu \subset W_{J_{\max}}.$$

Note by Proposition 5.4 that $W''_\mu = W_{\tilde{J}_{\max}^\mu}$, where \tilde{J}_{\max}^μ is the unique maximal set J' of simple roots (by Theorem A) such that $\text{wt}_{J \cap J(\mathbb{V}^\lambda)} L_{J(\mathbb{V}^\lambda)}(\eta_\mu) = \text{wt}_{J'} L_{J(\mathbb{V}^\lambda)}(\eta_\mu)$. Moreover, we claim – akin to the proof of Theorem A – that the inclusion in (5.5) holds even if the intersection is taken over the smaller set $\mu \in \{0\} \sqcup \Delta_{J_2(\mathbb{V}^\lambda)}$. In other words, the proof is complete if the following inclusion is shown to hold:

$$(5.6) \quad \tilde{J}_{\max}^0 \cap \bigcap_{j \in J_2(\mathbb{V}^\lambda)} \tilde{J}_{\max}^{\alpha_j} \subset J_{\max}.$$

To prove this inclusion, first note that $\tilde{J}_{\max}^0 = (J \cap J(\mathbb{V}^\lambda))_{\max} \supset J_{\max}$ (the inclusion follows from Equation (3.3)). We now claim that for all $j_6 \in (J \cap J(\mathbb{V}^\lambda))_{\max} \setminus J_{\max}$, there exists $j \in J_2(\mathbb{V}^\lambda)$ that $j_6 \notin \tilde{J}_{\max}^{\alpha_j}$. To show this claim, first compute using

equation (3.3):

$$(J \cap J(\mathbb{V}^\lambda))_{\max} \setminus J_{\max} = J_6(\mathbb{V}^\lambda) \cap \{\lambda\}^\perp \cap J_3(\mathbb{V}^\lambda)^\perp \setminus (J_2(\mathbb{V}^\lambda)^\perp \cap J_4(\mathbb{V}^\lambda)^\perp).$$

Thus it remains to consider two cases. The first is if $j_6 \in J_6(\mathbb{V}^\lambda) \cap \{\lambda\}^\perp \cap J_3(\mathbb{V}^\lambda)^\perp \setminus J_2(\mathbb{V}^\lambda)^\perp$. Suppose $(\alpha_{j_6}, \alpha_{j_2}) \neq 0$ for some $j_2 \in J_2(\mathbb{V}^\lambda)$. Then

$$s_{j_6}(\eta_{\alpha_{j_2}}) = s_{j_6}(\pi_{J(\mathbb{V}^\lambda)}(\lambda - \alpha_{j_2})) \in \text{wt}_{\{j_6\}} L_{J(\mathbb{V}^\lambda)}(\eta_{\alpha_{j_2}}) \setminus \text{wt}_{J \cap J(\mathbb{V}^\lambda)} L_{J(\mathbb{V}^\lambda)}(\eta_{\alpha_{j_2}}).$$

Therefore $j_6 \notin \tilde{J}_{\max}^{\alpha_{j_2}}$ as desired.

The second case is when $j_6 \in J_6(\mathbb{V}^\lambda) \cap \{\lambda\}^\perp \cap J_3(\mathbb{V}^\lambda)^\perp \cap J_2(\mathbb{V}^\lambda)^\perp \setminus J_4(\mathbb{V}^\lambda)^\perp$. Then there exist $j_2 \in J_2(\mathbb{V}^\lambda)$ and a connected component $C \subset J_4(\mathbb{V}^\lambda)$ of $J \cap J(\mathbb{V}^\lambda)$, such that neither α_{j_6} nor α_{j_2} is orthogonal to all of Δ_C . Say $(\alpha_{j_6}, \alpha_{j_4}) \neq 0$ for $j_4 \in C$. We assert that the affine hull of the set $S' := \text{wt}_{J \cap J(\mathbb{V}^\lambda)} L_{J(\mathbb{V}^\lambda)}(\eta_{\alpha_{j_2}})$ contains $\lambda - \mathbb{R}\Delta_C$; but this holds by definition of $J_3(\mathbb{V}^\lambda)$ and [KR, Proposition 5.1], since $\pi_C(\eta_{\alpha_{j_2}}) = \pi_C(\lambda - \alpha_{j_2}) \neq 0$. In particular, the difference $\eta_{\alpha_{j_2}} - w_o^{J \cap J(\mathbb{V}^\lambda)}(\eta_{\alpha_{j_2}})$ of the extremal elements in S' lies in $Q^+ \setminus Q_{I \setminus \{j_4\}}^+$. Therefore,

$$s_{j_6}(w_o^{J \cap J(\mathbb{V}^\lambda)}(\eta_{\alpha_{j_2}})) \in \text{wt}_{(J \cap J(\mathbb{V}^\lambda)) \sqcup \{j_6\}} L_{J(\mathbb{V}^\lambda)}(\eta_{\alpha_{j_2}}) \setminus \text{wt}_{J \cap J(\mathbb{V}^\lambda)} L_{J(\mathbb{V}^\lambda)}(\eta_{\alpha_{j_2}}),$$

which again shows that $j_6 \notin \tilde{J}_{\max}^{\alpha_{j_2}}$, proving the claim made after equation (5.6). Putting together the above analysis shows that (5.6) holds, whence $W_{J_{\max}} \subset W' \subset \bigcap_{\mu \in \mathbb{T}(\mathbb{V}^\lambda)} W''_\mu \subset W_{J_{\max}}$ as desired. Finally, that $W_{J_{\max}} = W_{J_{\min}} \times W_{J_{\max} \setminus J_{\min}}$ was proved in Proposition 5.4.

Next, suppose $\text{conv}_{\mathbb{R}} \text{wt } \mathbb{V}^\lambda = \text{conv}_{\mathbb{R}} \text{wt } M(\lambda, J(\mathbb{V}^\lambda))$. Then $\text{conv}_{\mathbb{R}} \text{wt}_J \mathbb{V}^\lambda$ is a face of the convex polyhedron $\text{conv}_{\mathbb{R}} \text{wt } \mathbb{V}^\lambda$ for all $J \subset I$. We now compute (the size of) the vertex set of this face. Using Theorems 2.5 and A, it follows that

$$\text{conv}_{\mathbb{R}} \text{wt}_J \mathbb{V}^\lambda = \text{conv}_{\mathbb{R}} \text{wt}_J M(\lambda) \cap \text{conv}_{\mathbb{R}} \text{wt } M(\lambda, J(\mathbb{V}^\lambda)) = \text{conv}_{\mathbb{R}} \text{wt}_J M(\lambda, J(\mathbb{V}^\lambda)).$$

Thus, applying Theorem 2.5 for the \mathfrak{g}_J -submodule $\mathbb{V}_J^\lambda := U(\mathfrak{g}_J)v_\lambda \subset M(\lambda, J(\mathbb{V}^\lambda))$ shows that the face $\text{conv}_{\mathbb{R}} \text{wt}_J \mathbb{V}^\lambda$ has vertex set $W_{J \cap J(\mathbb{V}^\lambda)}(\lambda)$. Now use Lemma 4.5 to reduce the problem to studying the vertex set inside the convex hull of weights of the finite-dimensional $\mathfrak{g}_{J(\mathbb{V}^\lambda)}$ -module $L_{J(\mathbb{V}^\lambda)}(\pi_{J(\mathbb{V}^\lambda)}(\lambda))$. By Proposition 4.1, the vertex set has size $|W_{J \cap J(\mathbb{V}^\lambda)}(\lambda)| = [W_{J \cap J(\mathbb{V}^\lambda)} : W'']$, where W'' is the stabilizer subgroup in $W_{J \cap J(\mathbb{V}^\lambda)}$ of $\pi_{J(\mathbb{V}^\lambda)}(\lambda)$. Since $\pi_{J(\mathbb{V}^\lambda)}(\lambda) \in P^+$, it follows by assertion (I) in [Bou, Chapter V.3.3] and Theorem 2 in [Bou, Chapter VI.1.5] that W'' is generated by the simple reflections s_j in it. Now s_j fixes $\pi_{J(\mathbb{V}^\lambda)}(\lambda)$ for $j \in J \cap J(\mathbb{V}^\lambda)$, if and only if $j \in J \cap J(\mathbb{V}^\lambda) \cap \{\pi_{J(\mathbb{V}^\lambda)}(\lambda)\}^\perp$. It follows that $W'' = W_{J \cap J(\mathbb{V}^\lambda) \cap \{\pi_{J(\mathbb{V}^\lambda)}(\lambda)\}^\perp}$, and the proof is complete upon noting that $J(\mathbb{V}^\lambda) \cap \{\pi_{J(\mathbb{V}^\lambda)}(\lambda)\}^\perp = J(\mathbb{V}^\lambda) \cap \{\lambda\}^\perp$.

Finally, to compute the f -polynomial of the convex polyhedron $\text{conv}_{\mathbb{R}} \text{wt } \mathbb{V}^\lambda$, apply Theorem 2.5 for $\mathbb{V}^\lambda = M(\lambda, J(\mathbb{V}^\lambda))$ to obtain that every face of the convex polyhedron $\text{conv}_{\mathbb{R}} \text{wt } \mathbb{V}^\lambda$ is $W_{J(\mathbb{V}^\lambda)}$ -conjugate to a unique face of the form $\text{conv}_{\mathbb{R}} \text{wt}_{J_{\max}} \mathbb{V}^\lambda$ (or $\text{conv}_{\mathbb{R}} \text{wt}_{J_{\min}} \mathbb{V}^\lambda$). The result now follows from the first two parts of this theorem. \square

Remark 5.7. Note by the analysis in the penultimate paragraph of the proof of Theorem B that $[W_J : W_{J \cap \{\pi_{J(\mathbb{V}^\lambda)}(\lambda)\}^\perp}]$ does not depend on the choice of the set J from among $[J_{\min}, J_{\max}]$. This can also be seen directly as follows: recall that

$W_{J_{\max} \setminus J_{\min}}$ fixes λ and commutes with $W_{J_{\min}}$, so that $J \cap J(\mathbb{V}^\lambda) \setminus J_{\min} \subset \{\lambda\}^\perp$. Therefore,

$$\begin{aligned} & [W_{J \cap J(\mathbb{V}^\lambda)} : W_{J \cap J(\mathbb{V}^\lambda) \cap \{\lambda\}^\perp}] \\ &= [W_{J_{\min} \sqcup (J \cap J(\mathbb{V}^\lambda) \setminus J_{\min})} : W_{(J_{\min} \cap \{\lambda\}^\perp) \sqcup (J \cap J(\mathbb{V}^\lambda) \setminus J_{\min})}] \\ &= [W_{J_{\min}} \times W_{J \cap J(\mathbb{V}^\lambda) \setminus J_{\min}} : W_{J_{\min} \cap \{\lambda\}^\perp} \times W_{J \cap J(\mathbb{V}^\lambda) \setminus J_{\min}}] \\ &= [W_{J_{\min}} : W_{J_{\min} \cap \{\lambda\}^\perp}], \end{aligned}$$

and this is indeed independent of $J \in [J_{\min}, J_{\max}]$. We also remark that Theorem B yields multiple formulas for the number of vertices of $\text{conv}_{\mathbb{R}} \text{wt } \mathbb{V}^\lambda$, and these formulas are easily seen to agree in light of the preceding computation.

We end this section by discussing how a result of Satake, Borel–Tits, Vinberg, and Casselman for Weyl polytopes with generic dominant integral highest weight $\lambda \in P^+$ follows from Theorems A and B. These authors showed that the distinct sets among $\{\text{wt}_J L(\lambda) : J \subset I\}$ are in bijection with “ λ -admissible” subsets $J \subset I$, when $\lambda \in P^+$ is “admissible”. In the notation of the present paper, $\mathbb{V}^\lambda = M(\lambda, I) = L(\lambda)$ and $J(L(\lambda)) = I$. Moreover, it is not hard to verify that λ being admissible simply means that $I_{\min} = I$, while $J \subset I$ being λ -admissible means that $J = J_{\min} = J_3(L(\lambda))$. We now write down the aforementioned result, as stated by Vinberg.

Theorem 5.8 (Vinberg, [Vi, Proposition 3.2]). *Fix $\lambda \in P^+$ and $\mathbb{V}^\lambda = L(\lambda)$. Suppose $I_{\min} = I$. Then every face of $\mathcal{P}(\lambda) = \text{conv}_{\mathbb{R}} \text{wt } L(\lambda)$ is W -conjugate to $\text{conv}_{\mathbb{R}} \text{wt}_J L(\lambda)$ for a unique subset $J \subset I$ such that $J = J_3(L(\lambda)) = J_{\min}$. Moreover, this face has dimension $|J_{\min}|$.*

Note that Theorem 5.8 follows from our main Theorems A and B. In fact these two Theorems show that the assumption $I_{\min} = I$ is not required to prove Theorem 5.8. (This assumption was also used in [Vi] to ensure that the Weyl polytope is of “full dimension” $|I_{\min}| = |I|$.) As discussed in Remark 3.9, the analysis in Theorems A and B for general modules \mathbb{V}^λ is more involved because one has to account for the simple roots in $J_2(\mathbb{V}^\lambda)$ and hence in $J_4(\mathbb{V}^\lambda)$.

6. HALF-SPACE REPRESENTATION AND FACETS

We now study standard parabolic subsets of weights in $\text{wt } \mathbb{V}^\lambda$, whose convex or affine hull has codimension one in $\text{conv}_R \text{wt } \mathbb{V}^\lambda$. The goal in this section is to prove Theorem C, using the fact that every convex polyhedron is the intersection of a minimal family of codimension one facets. In order to do so, it is natural to seek characterizations of when a particular face $\text{conv}_{\mathbb{R}} \text{wt}_{I \setminus \{i\}} \mathbb{V}^\lambda$ is a (codimension-one) facet of $\text{conv}_{\mathbb{R}} \text{wt } \mathbb{V}^\lambda$. The following result provides several different such characterizations, for all highest weight modules \mathbb{V}^λ .

Proposition 6.1. *Suppose $\lambda \in \mathfrak{h}^*$ and $M(\lambda) \twoheadrightarrow \mathbb{V}^\lambda$. Given $i \in I$, define the “coordinate face” $F_i(\mathbb{V}^\lambda) := \text{conv}_{\mathbb{R}} \text{wt}_{I \setminus \{i\}} \mathbb{V}^\lambda$. Then the following are equivalent for $i \in I$:*

- (1) $F_i(\mathbb{V}^\lambda)$ has codimension one in the real affine space $\lambda - \mathbb{R}\Delta_{I_{\min} \sqcup (I \setminus J(\mathbb{V}^\lambda))}$.
- (2) $i \notin J(\mathbb{V}^\lambda) \setminus I_{\min} = I_{\max} \setminus I_{\min}$, and $F_i(\mathbb{V}^\lambda)$ is maximal among the coordinate faces $\{F_{i'}(\mathbb{V}^\lambda) : i' \in I_{\min} \sqcup (I \setminus J(\mathbb{V}^\lambda))\}$.
- (3) $i \notin J(\mathbb{V}^\lambda) \setminus I_{\min}$, and for all $i' \in I_{\min} \setminus \{i\}$, there exists $\mu \in \{0\} \sqcup \Delta_{I \setminus J(\mathbb{V}^\lambda)}$ such that the minimal weight μ_i of $\text{wt}_{I_{\min} \setminus \{i\}} L_{J(\mathbb{V}^\lambda)}(\pi_{J(\mathbb{V}^\lambda)}(\lambda - \mu))$ satisfies: $(\mu_i, \omega_{i'}) \neq (\lambda, \omega_{i'})$.

- (4) $(I \setminus \{i\})_{\min} = I_{\min} \setminus \{i\}$ and $(I \setminus \{i\})_{\max} = I_{\max} \setminus \{i\}$.
- (5) $i \notin J(\mathbb{V}^\lambda) \setminus I_{\min}$, and for all $i' \in I_{\min} \setminus \{i\}$, there exists $\mu \in \text{wt}_{I \setminus \{i\}} \mathbb{V}^\lambda$ such that $(\mu, \omega_{i'}) \neq (\lambda, \omega_{i'})$.
- (6) $i \notin J(\mathbb{V}^\lambda) \setminus I_{\min}$, and for all $i' \in I_{\min} \setminus \{i\}$, the set $\text{wt}_{I \setminus \{i\}} \mathbb{V}^\lambda$ contains a nontrivial $\alpha_{i'}$ -string.

We observe (via the dictionary in Section 8) that [CM, Theorem 4.5] is a special case of Proposition 6.1, where \mathfrak{g} is simple and \mathbb{V}^λ is the adjoint representation. More precisely, each part of Proposition 6.1 corresponds to the same numbered part in *loc. cit.*, with the exception of the second part above, which corresponds to part (7) of *loc. cit.* (For part (3), recall the second equality in equation (3.2).) We will address the missing part [CM, Theorem 4.5(2)] in Corollary 8.6 below.

Note by Theorem A that $I_{\max} = J(\mathbb{V}^\lambda)$ by maximality. Therefore Proposition 6.1(4) reads:

$$(6.2) \quad (I \setminus \{i\})_{\min} = I_{\min} \setminus \{i\}, \quad J(\mathbb{V}^\lambda) \setminus \{i\} = (I \setminus \{i\})_{\max}.$$

The second equation in (6.2) is *a priori* different from what appears in four of the six assertions in Proposition 6.1 – namely, the condition $i \notin J(\mathbb{V}^\lambda) \setminus I_{\min}$. We now show a preliminary result which is required to prove Proposition 6.1, and which shows that the aforementioned two conditions on i are equivalent for all highest weight modules \mathbb{V}^λ .

Proposition 6.3. *Suppose $\lambda \in \mathfrak{h}^*$ and $M(\lambda) \twoheadrightarrow \mathbb{V}^\lambda$. Then for all $i \in I$,*

$$(I \setminus \{i\})_{\max} = \begin{cases} J(\mathbb{V}^\lambda) \setminus \{i\} & \text{if } i \in I_{\min} \sqcup (I \setminus J(\mathbb{V}^\lambda)), \\ J(\mathbb{V}^\lambda) & \text{otherwise, i.e., if } i \in J(\mathbb{V}^\lambda) \setminus I_{\min} = I_{\max} \setminus I_{\min}. \end{cases}$$

Moreover, the “coordinate faces” $\{\text{wt}_{I \setminus \{i\}} \mathbb{V}^\lambda : i \in I_{\min} \sqcup (I \setminus J(\mathbb{V}^\lambda))\}$ are distinct and proper subsets of $\text{wt} \mathbb{V}^\lambda$, which equals $\text{wt}_{I \setminus \{i\}} \mathbb{V}^\lambda$ for all $i \in J(\mathbb{V}^\lambda) \setminus I_{\min}$.

Note that the second assertion extends [CM, Proposition 4.1] from the adjoint representation (for simple \mathfrak{g}) to all highest weight modules.

Proof. If $i \in I \setminus J(\mathbb{V}^\lambda)$, then $(I \setminus \{i\})_{\max} = J(\mathbb{V}^\lambda) = J(\mathbb{V}^\lambda) \setminus \{i\}$ by Theorem A. If $i \in I_{\min}$, then it follows from Theorem A that $\text{wt}_I \mathbb{V}^\lambda = \text{wt} \mathbb{V}^\lambda \neq \text{wt}_{I \setminus \{i\}} \mathbb{V}^\lambda$, by taking the intersection with $\text{wt}_{J(\mathbb{V}^\lambda)} \mathbb{V}^\lambda$. Therefore $(I \setminus \{i\})_{\max} \neq J(\mathbb{V}^\lambda)$ as desired. Now suppose $i \in J(\mathbb{V}^\lambda) \setminus I_{\min} = I_{\max} \setminus I_{\min}$. Then $\text{wt}_I \mathbb{V}^\lambda = \text{wt}_{I_{\min} \sqcup (I \setminus J(\mathbb{V}^\lambda))} \mathbb{V}^\lambda$, so the intermediate set $\text{wt}_{I \setminus \{i\}} \mathbb{V}^\lambda$ also equals $\text{wt} \mathbb{V}^\lambda$. It follows that $(I \setminus \{i\})_{\max} = J(\mathbb{V}^\lambda)$, proving the formula.

Next, suppose $\text{wt}_{I \setminus \{i_1\}} \mathbb{V}^\lambda = \text{wt}_{I \setminus \{i_2\}} \mathbb{V}^\lambda = S'$, say, where $i_1 \neq i_2$. By Theorem A, $i_1, i_2 \in J(\mathbb{V}^\lambda)$ and there is a maximal set $J_{\max} \subset I$ such that $S' = \text{wt}_{J_{\max} \sqcup (I \setminus J(\mathbb{V}^\lambda))} \mathbb{V}^\lambda$. In particular, $J_{\max} \supset J(\mathbb{V}^\lambda) \setminus \{i_l\}$ for $l = 1, 2$, whence $J_{\max} = J(\mathbb{V}^\lambda)$. Thus $\text{wt}_{I \setminus \{i_l\}} \mathbb{V}^\lambda = \text{wt}_{J_{\max} \sqcup (I \setminus J(\mathbb{V}^\lambda))} \mathbb{V}^\lambda = \text{wt} \mathbb{V}^\lambda$, so that $(I \setminus \{i_l\})_{\max} = J(\mathbb{V}^\lambda)$. The second assertion now follows from the preceding paragraph. \square

We can now prove the above characterization of facets in highest weight modules.

Proof of Proposition 6.1. We first show that $(1) \implies (2) \implies (6) \implies (5) \implies (4) \implies (1)$. Recall from Theorem B(1) that the affine hull of $\text{wt} \mathbb{V}^\lambda$ is $\lambda - \mathbb{R}\Delta_{I_{\min} \sqcup (I \setminus J(\mathbb{V}^\lambda))}$. Now if (1) holds, then note by Proposition 6.3 that $i \notin J(\mathbb{V}^\lambda) \setminus I_{\min}$, and also that if $i' \in I_{\min} \sqcup (I \setminus J(\mathbb{V}^\lambda))$ then $F_{i'}(\mathbb{V}^\lambda) \subsetneq \text{wt} \mathbb{V}^\lambda$. Thus if $F_i(\mathbb{V}^\lambda) \subset F_{i'}(\mathbb{V}^\lambda)$ then they are equal. Intersecting with $\text{wt} \mathbb{V}^\lambda$ yields: $\text{wt}_{I \setminus \{i\}} \mathbb{V}^\lambda =$

$\text{wt}_{I \setminus \{i'\}} \mathbb{V}^\lambda$, which contradicts Proposition 6.3 if $i' \neq i$. Thus (1) \implies (2). Now assume that (6) fails; then $\text{wt}_{I \setminus \{i\}} \mathbb{V}^\lambda$ does not contain a nontrivial $\alpha_{i'}$ -string or an α_i -string. Therefore

$$\text{wt}_{I \setminus \{i\}} \mathbb{V}^\lambda = \text{wt}_{I \setminus \{i, i'\}} \mathbb{V}^\lambda \subset \text{wt}_{I \setminus \{i'\}} \mathbb{V}^\lambda$$

by Lemma 4.4, and the inclusion is strict by Proposition 6.3. This contradicts (2), whence (2) \implies (6). Clearly (6) \implies (5) since $\text{wt} \mathbb{V}^\lambda \subset \lambda - \mathbb{Z}_+ \Delta$. We show next that (5) \implies (4). First observe from (5) that every $i' \in I_{\min} \setminus \{i\}$ is continued in $(I \setminus \{i\})_{\min}$. It remains to show the reverse inclusion that $(I \setminus \{i\})_{\min} \subset I_{\min} \setminus \{i\}$. To show this, note by (5) that there are two cases: first if $i \in I \setminus J(\mathbb{V}^\lambda)$, then $(I \setminus \{i\})_{\min} \subset I_{\min} = I_{\min} \setminus \{i\}$, as desired. The other case is if $i \in I_{\min}$. Now compute using Theorem A:

$$\begin{aligned} \text{wt}_{(I \setminus \{i\})_{\min} \sqcup (I \setminus (\{i\} \cup J(\mathbb{V}^\lambda)))} \mathbb{V}^\lambda &= \text{wt}_{I \setminus \{i\}} \mathbb{V}^\lambda = \text{wt}_{I \setminus \{i\}} \mathbb{V}^\lambda \cap \text{wt} \mathbb{V}^\lambda \\ &= \text{wt}_{(J(\mathbb{V}^\lambda) \setminus \{i\}) \sqcup (I \setminus (\{i\} \cup J(\mathbb{V}^\lambda)))} \mathbb{V}^\lambda \cap \text{wt}_{I_{\min} \sqcup (I \setminus J(\mathbb{V}^\lambda))} \mathbb{V}^\lambda = \text{wt}_{(I_{\min} \setminus \{i\}) \sqcup (I \setminus J(\mathbb{V}^\lambda))} \mathbb{V}^\lambda. \end{aligned}$$

Again using Theorem A, it follows that $(I \setminus \{i\})_{\min} \subset I_{\min} \setminus \{i\}$. This shows that (5) \implies (4). Now if (4) holds, then compute using Theorem B(1):

$$\begin{aligned} \dim F_i(\mathbb{V}^\lambda) &= |(I \setminus \{i\})_{\min} \sqcup ((I \setminus \{i\}) \setminus J(\mathbb{V}^\lambda))| \\ &= |(J(\mathbb{V}^\lambda) \setminus \{i\}) \sqcup ((I \setminus \{i\}) \setminus J(\mathbb{V}^\lambda))| = |I \setminus \{i\}|, \end{aligned}$$

where the last equality follows from (4). Thus (4) \implies (1).

It remains to show that (3) is equivalent to the other assertions. Via Lemma 4.5 we will identify $\text{wt} L_{J(\mathbb{V}^\lambda)}(\pi_{J(\mathbb{V}^\lambda)}(\lambda - \mu))$ with $\text{wt} L_{J(\mathbb{V}^\lambda)}(\lambda - \mu)$. We now show that (4) \implies (3) \implies (1). First recall from above that $(I \setminus \{i\})_{\max} = I_{\max} \setminus \{i\}$ is equivalent to $i \notin J(\mathbb{V}^\lambda) \setminus I_{\min}$. Now suppose (4) holds, i.e., $(I \setminus \{i\})_{\min} = I_{\min} \setminus \{i\}$. If $i' \in I_{\min} \setminus i$, then there exists a connected component C of the Dynkin diagram of $I_{\min} \setminus \{i\}$ such that $i' \in C$. By equation (3.2), if $\pi_C(\lambda) = 0 \neq (\Delta_C, \Delta_{I \setminus J(\mathbb{V}^\lambda)})$, then set $\mu := \alpha_{i_2}$ for $i_2 \in I \setminus J(\mathbb{V}^\lambda)$ such that $(\Delta_C, \alpha_{i_2}) \neq 0$; while if $\pi_C(\lambda) = 0$ then choose $\mu := 0$. It follows that $i' \in C \subset K_3(\lambda - \mu)$, where $K = I_{\min} \setminus \{i\}$. Now apply the minimality of $I_{\min} \setminus \{i\}$ by (4), as well as [KR, Proposition 5.1], to the finite-dimensional module $L_{J(\mathbb{V}^\lambda)}(\lambda - \mu)$. Thus, there is at least one weight μ'_i such that $\lambda - \mu'$ is a sum of simple roots with at least one simple root equal to $\alpha_{i'}$. In particular, for each $i' \in I_{\min} \setminus \{i\}$ it follows that $(\mu_i, \omega_{i'}) \neq (\lambda, \omega_{i'})$, proving (3).

Finally, suppose (3) holds. Note that the affine hull of $\text{conv}_{\mathbb{R}} \text{wt}_{I \setminus \{i\}} \mathbb{V}^\lambda$ contains $\Delta_{I \setminus (\{i\} \cup J(\mathbb{V}^\lambda))}$ by definition of $J(\mathbb{V}^\lambda)$. Next, given $i' \in I_{\min} \setminus \{i\}$, it follows, using the notation of (3), that $\text{wt}_{I_{\min} \setminus \{i\}} L_{J(\mathbb{V}^\lambda)}(\lambda - \mu) \subset \text{wt} \mathbb{V}^\lambda$. It follows by (3) that $\Delta_{I_{\min} \setminus \{i\}}$ is also contained in the affine hull of $\text{conv}_{\mathbb{R}} \text{wt}_{I \setminus \{i\}} \mathbb{V}^\lambda$. Therefore (3) \implies (1) and the proof is complete. \square

Finally, we use the analysis in this and previous sections to prove our last main result.

Proof of Theorem C. Since $\text{wt} \mathbb{V}^\lambda \subset \lambda - \mathbb{Z}_+ \Delta$ and since $\text{wt} \mathbb{V}^\lambda$ is $W_{J(\mathbb{V}^\lambda)}$ -stable by Theorem 2.3, hence $\text{conv}_{\mathbb{R}} \text{wt} \mathbb{V}^\lambda \subset \bigcap_{i \in I, w \in W_{J(\mathbb{V}^\lambda)}} H_{i, w}$. Also note that $W_{(I \setminus \{i\})_{\max}}$ preserves the half-space $H_{i, 1}$ as well as its boundary, since $\text{wt}_{I \setminus \{i\}} \mathbb{V}^\lambda$ is contained in the boundary. It follows that the above intersection remains unchanged even if it runs only over $\{W^i : i \in I\}$.

Now since it is also known that $\text{conv}_{\mathbb{R}} \text{wt } \mathbb{V}^\lambda$ is a convex polyhedron, it equals its minimal half-space representation, i.e., the intersection of the half-spaces $H_{i,w}$ corresponding to the codimension-one facets of $\text{conv}_{\mathbb{R}} \text{wt } \mathbb{V}^\lambda$. Note here that the codimension of the faces of $\text{conv}_{\mathbb{R}} \text{wt } \mathbb{V}^\lambda$ is computed inside its affine hull, which is $\lambda - \mathbb{R}\Delta_{I_{\min} \sqcup (I \setminus J(\mathbb{V}^\lambda))}$ by Theorem B(1). By Theorem 2.5 for $\mathbb{V}^\lambda = M(\lambda, J(\mathbb{V}^\lambda))$, the codimension-one faces of $\text{conv}_{\mathbb{R}} \text{wt } \mathbb{V}^\lambda$ correspond to $i \in I$ (and any w) such that the supporting hyperplane of $H_{i,1}$, which is the affine hull of $\text{conv}_{\mathbb{R}} \text{wt}_{I \setminus \{i\}} \mathbb{V}^\lambda$, has codimension one. By Proposition 6.1, this condition is equivalent to: $i \in I_{\min} \sqcup (I \setminus J(\mathbb{V}^\lambda))$, and $(I \setminus \{i\})_{\min} = I_{\min} \setminus \{i\}$. Thus the second part of the assertion is proved, and hence the first part as well. \square

7. MINIMUM ELEMENTS AND LONGEST WEIGHTS IN COMPACT FACES

In this section we present additional results on minimum elements in “compact” faces (i.e., ones containing only finitely many weights) of $\text{conv}_{\mathbb{R}} \text{wt } \mathbb{V}^\lambda$. We also show that there is a natural analogue in any highest weight module \mathbb{V}^λ , of the long roots in the adjoint representation. The following result characterizes the standard parabolic subsets of highest weight modules that contain minimal elements, and also identifies these elements as well as the longest weights.

Proposition 7.1. *Fix $\lambda \in \mathfrak{h}^*$, $M(\lambda) \rightarrow \mathbb{V}^\lambda$, and $J \subset I$. Then the following are equivalent.*

- (1) $\text{wt}_J \mathbb{V}^\lambda$ has a longest element (i.e., a weight μ with maximum Euclidean norm (μ, μ)).
- (2) $\text{conv}_{\mathbb{R}} \text{wt}_J \mathbb{V}^\lambda$ is a convex polytope.
- (3) $J \subset J(\mathbb{V}^\lambda)$.
- (4) $\text{wt}_J \mathbb{V}^\lambda$ has a minimum element in the standard partial order on \mathfrak{h}^* .

If these conditions hold, then the longest weights in $\text{wt}_J \mathbb{V}^\lambda$ are precisely $W_J(\lambda)$. These include the maximum and minimum elements in $\text{wt}_J \mathbb{V}^\lambda$ in the standard partial order, which are unique and equal λ and $w_\circ^J(\lambda)$ respectively.

Proof. If (1) holds, then $\text{wt}_J \mathbb{V}^\lambda$ cannot contain any string of the form $\lambda - \mathbb{Z}_+ \alpha_i$ for $i \in I \setminus J(\mathbb{V}^\lambda)$, so (3) follows. Clearly (3) \implies (2) using Theorem 2.3. Now if (2) holds then $\text{wt}_J \mathbb{V}^\lambda$ is a finite set, hence has a longest element. Thus (1), (2), and (3) are equivalent. Next if $j \in J \setminus J(\mathbb{V}^\lambda)$ then $\lambda - \mathbb{Z}_+ \alpha_j \subset \text{wt}_J \mathbb{V}^\lambda$. Therefore (4) \implies (3). Conversely, if $J \subset J(\mathbb{V}^\lambda)$ then $\varpi_J : \text{wt}_J \mathbb{V}^\lambda \rightarrow \text{wt } L_J(\pi_J(\lambda))$ is a bijection by Lemma 4.5. Therefore it has a minimum element $w_\circ^J(\lambda)$, which is unique by lowest weight theory. Hence (1)–(4) are equivalent.

Finally, note that every standard parabolic subset $\text{wt}_J \mathbb{V}^\lambda$ always contains the unique maximum element λ . Now if $J \subset I$ is such that $\text{conv}_{\mathbb{R}} \text{wt}_J \mathbb{V}^\lambda$ is a convex, compact polytope, the norm function attains its maximum value on this polytope. It is easy to verify that the norm cannot be maximized at an interior point of a line segment. Therefore the maximum value is attained at a vertex, i.e., at a point in $W_{J \cap J(\mathbb{V}^\lambda)}(\lambda) = W_J(\lambda)$ (by Proposition 4.1). The proof is completed by recalling that $W_{J(\mathbb{V}^\lambda)} \subset W$ acts on \mathfrak{h}^* by isometries. \square

It is also possible to obtain characterizations of the sets $J_3(\mathbb{V}^\lambda)$, as well as of the minimal weights in compact faces of highest weight modules. The following result, together with Proposition 7.1, accomplishes these goals.

Proposition 7.2. Fix $\lambda \in \mathfrak{h}^*$ and $M(\lambda) \twoheadrightarrow \mathbb{V}^\lambda$. Given $\mu \in \text{wt } M(\lambda)$, define

$$(7.3) \quad I_\lambda(\mu) := \{i \in I : (\lambda - \mu, \omega_i) = 0\}.$$

- (1) Given $J \subset I$, $J = J_3(\mathbb{V}^\lambda)$ if and only if $J = J(\mathbb{V}^\lambda) \setminus I_\lambda(\mu)$ for some $\mu \in \text{wt}_{J(\mathbb{V}^\lambda)} \mathbb{V}^\lambda$.
- (2) Suppose $\mu \in \text{wt}_{J(\mathbb{V}^\lambda)} \mathbb{V}^\lambda$. Then $\mu = \min(\text{wt}_{J(\mathbb{V}^\lambda) \setminus I_\lambda(\mu)} \mathbb{V}^\lambda)$ if and only if μ is a longest weight in $\text{wt}_{J(\mathbb{V}^\lambda)} \mathbb{V}^\lambda$ and $(\mu, \alpha_i) \leq 0$ for all $i \in J(\mathbb{V}^\lambda) \setminus I_\lambda(\mu)$.

In particular, the result provides a characterization of the sets $J_{\min}(\mathbb{V}^\lambda)$ for all finite-dimensional modules $\mathbb{V}^\lambda = M(\lambda, I) = L(\lambda)$ when $\lambda \in P^+$.

Note that the $W_{J(\mathbb{V}^\lambda)}$ -orbit of all “longest weights” in $\text{wt}_{J(\mathbb{V}^\lambda)} \mathbb{V}^\lambda$ is a generalization in arbitrary \mathbb{V}^λ of the W -orbit of long roots in the adjoint representation. Apart from the length, the longest weights generalize to \mathbb{V}^λ several properties satisfied by the long roots in $L(\theta) = \mathfrak{g}$ for simple \mathfrak{g} . For instance, Propositions 7.1 and 7.2 extend [CM, Proposition 3.3(1), Remark 3.8, Lemma 3.12, and Propositions 3.9 and 3.16] to all highest weight modules \mathbb{V}^λ .

Proof. (1) It is not hard to compute from the definitions that $\lambda - w_\circ^J(\lambda) = \sum_{j \in J_3(\mathbb{V}^\lambda)} c_j \alpha_j$ with all $c_j > 0$. The formula (3.2) for $J = J_3(\mathbb{V}^\lambda)$ follows by using $\mu = w_\circ^J(\lambda)$ and the definition of $I_\lambda(\mu)$. Conversely, suppose $J = J(\mathbb{V}^\lambda) \setminus I_\lambda(\mu)$ for some $\mu \in \text{wt}_{J(\mathbb{V}^\lambda)} \mathbb{V}^\lambda$. Note by the definition of $I_\lambda(\mu)$ that $\mu \in \text{wt}_J \mathbb{V}^\lambda$ but $\mu \notin \text{wt}_{J'} \mathbb{V}^\lambda$ for any $J' \subsetneq J$. This minimality of J implies by Theorem A that $J = J_3(\mathbb{V}^\lambda)$.

(2) Set $J'_\mu := J(\mathbb{V}^\lambda) \setminus I_\lambda(\mu)$ for ease of exposition. If $\mu = \min(\text{wt}_{J'_\mu} \mathbb{V}^\lambda)$, then $\mu \in W_{J(\mathbb{V}^\lambda)}(\lambda)$, whence μ is a longest weight. Moreover, μ is the lowest weight of the $\mathfrak{g}_{J'_\mu}$ -module $L_{\mathfrak{g}_{J'_\mu}}(\lambda)$, so $s_i(\mu) \geq \mu$ for $i \in J'_\mu$. This implies that $(\mu, \alpha_i) \leq 0$ for $i \in J'_\mu$. To prove the converse, first note that since μ is a longest weight, it is of the form $W_{J(\mathbb{V}^\lambda)}(\lambda)$. Moreover, $\mu \in \text{wt}_{J'_\mu} \mathbb{V}^\lambda$ (as in the previous part) by definition of $I_\lambda(\mu)$. Therefore $\mu \in W_{J(\mathbb{V}^\lambda)}(\lambda) \cap \text{wt}_{J'_\mu} \mathbb{V}^\lambda = W_{J'_\mu}(\lambda)$. The remainder of the argument follows the proof of [CM, Proposition 3.16]. \square

8. EXAMPLE: FINITE-DIMENSIONAL REPRESENTATIONS OVER A SIMPLE LIE ALGEBRA

For completeness, we conclude this paper by pointing out several connections between the results proved in this paper, and results of the recent paper [CM] by Cellini and Marietti as well as the subsequent preprint [LCL] by Li–Cao–Li. Throughout this section, let \mathfrak{g} denote a complex simple Lie algebra.

We first discuss the work [CM], in which the authors study the faces of the root polytope $\mathcal{P}(\theta) := \text{conv}_{\mathbb{R}} \Phi$ for \mathfrak{g} . This corresponds to the special case where $\lambda = \theta$ is the highest root and $\mathbb{V}^\lambda = M(\theta, I) = L(\theta) = \mathfrak{g}$. It is not hard to see in this case that

$$(8.1) \quad J_\lambda = J(\mathbb{V}^\lambda) = I_{\max} = I = I_{\min}$$

(since $\theta \neq 0$), and that F_J equals $\text{conv}_{\mathbb{R}} \text{wt}_{I \setminus J} L(\theta) = \text{conv}_{\mathbb{R}} \text{wt}_{I \setminus J} \mathfrak{g}$ in our notation. Now given $J \subset I$, [CM, Proposition 3.7] says that $F_{J'} = F_J$ if and only if $\partial J \subset J' \subset \overline{J}$. On the other hand, Theorem A implies in the special case $\mathbb{V}^\lambda = M(\theta, I) = \mathfrak{g}$ that

$$\text{wt}_{J'} \mathfrak{g} = \text{wt}_J \mathfrak{g} \iff J_{\min} \subset J' \subset J_{\max}.$$

Given the uniqueness of the sets $\partial J, \overline{J}, J_{\min}, J_{\max} \subset I$ for every $J \subset I$, it is now possible to provide a dictionary between our notation and that used in [CM].

Proposition 8.2. *Suppose $\mathbb{V}^\lambda = \mathfrak{g}$ with \mathfrak{g} a simple Lie algebra, and $J \subset I$ is arbitrary. Then,*

$$(8.3) \quad \partial J = I \setminus (I \setminus J)_{\max}, \quad \overline{J} = I \setminus (I \setminus J)_{\min},$$

or conversely, $J_{\min} = I \setminus \overline{(I \setminus J)}$ and $J_{\max} = I \setminus \partial(I \setminus J)$.

Note that it is not hard to formulate equation (8.3), by comparing Proposition 6.1(4) and [CM, Theorem 4.5(4)], as well as from the formula for $(I \setminus \{i\})_{\max}$ in Proposition 6.3.

Proof. The second assertion follows from equation (8.3). To prove the first assertion, we first recall the definition of ∂J and \overline{J} from [CM, Section 3]. Let $\widehat{I} := I \sqcup \{0\}$ correspond to the simple roots $\widehat{\Delta} := \Delta \sqcup \{\alpha_0\}$ associated to the affine Lie algebra $\widehat{\mathfrak{g}}$. Now given $J \subset I$, let $(\widehat{I} \setminus J)_0$ denote the connected component of the Dynkin diagram of $\widehat{I} \setminus J$ that contains the affine root α_0 , and define:

$$(8.4) \quad \overline{J} := I \setminus (\widehat{I} \setminus J)_0, \quad \partial J := J \setminus (\widehat{I} \setminus J)_0^\perp.$$

To see why (the first formula in) (8.3) follows from equations (3.2), (3.3), and (8.4), first recall from [Bou, Chapter VI.4.3] that in the Dynkin diagram for $\widehat{\mathfrak{g}}$, the affine root α_0 can be viewed as the negative of the highest root for simple \mathfrak{g} : $\alpha_0 = -\theta$. Thus the connected components C of (the Dynkin diagram of) $I \setminus J$ such that $\pi_C(\theta) \neq 0$ correspond precisely to those simple roots $\alpha_i \in \Delta_{I \setminus J}$ such that $\alpha_0(h_i) = -\theta(h_i) \neq 0$, i.e., such that C is contained in the connected component of $\widehat{I} \setminus J$ containing α_0 . Therefore $(I \setminus J)_3(L(\theta)) \sqcup \{0\} = \{\alpha_0\} \sqcup \Delta_{(I \setminus J)_3(\mathfrak{g})} = (\widehat{I} \setminus J)_0$ with a slight abuse of notation. Moreover, equations (8.1) and (3.2) imply that $K_{\min} = K_3(\mathfrak{g})$ for all $K \subset I$. Therefore,

$$I \setminus (I \setminus J)_{\min} = I \setminus (I \setminus J)_3(\mathfrak{g}) = I \setminus (\widehat{I} \setminus J)_0 = \overline{J}.$$

This proves the first formula in (8.3). To show the second, we study equation (3.3) in closer detail in the current special case. By equation (8.1), $K_2(\mathfrak{g})$ and $K_4(\mathfrak{g})$ are empty for all $K \subset I$, whence $K_2(\mathfrak{g})^\perp = I$. Therefore using equation (3.3) and the above analysis for \overline{J} ,

$$\begin{aligned} I \setminus (I \setminus J)_{\max} &= I \setminus ((I \setminus J) \sqcup (J \cap \{\theta\}^\perp \cap (I \setminus J)_{\min}^\perp \cap I)) \\ &= J \setminus (\{\theta\}^\perp \cap (I \setminus J)_3(\mathfrak{g})^\perp) \\ &= J \setminus (\{\alpha_0\} \sqcup \Delta_{(I \setminus J)_3(\mathfrak{g})})^\perp = J \setminus (\widehat{I} \setminus J)_0^\perp = \partial J, \end{aligned}$$

which proves the second formula in (8.3). \square

Remark 8.5. The dictionary of Proposition 8.2 immediately helps translate the results in this paper, in the special case $\lambda = \theta$ and $\mathbb{V}^\lambda = L(\theta) = \mathfrak{g}$, into many results in [CM]. In particular, it follows that most of the results in Sections 1, 3, 4, and 5 of [CM] are specific manifestations of representation-theoretic phenomena that occur for all highest weight modules over all semisimple Lie algebras. Moreover, in [CM] the authors worked with the root system – i.e., the adjoint representation – and hence were able to prove their results using purely combinatorial arguments. In contrast, because we work with arbitrary highest weight modules, the present paper

provides an alternative, representation-theoretic approach to proving the results in [CM] for $L(\theta) = \mathfrak{g}$.

A further addition to the dictionary of Proposition 8.2 involves observing that many of the results in [CM] are stated in terms of the affine root α_0 and the affine root system $\widehat{\Phi}$ corresponding to the simple Lie algebra \mathfrak{g} . As noted above, α_0 is simply the negative of the highest root θ of \mathfrak{g} , i.e., the highest weight of the adjoint representation.

There are also certain results in [CM] that do not hold for all modules \mathbb{V}^λ , but are specific to the combinatorics of the root system, i.e., the adjoint representation $\mathfrak{g} = L(\theta)$. For instance, the size of the set $V_J = \text{wt}_{I \setminus J} L(\theta)$ is a computation specific to the root system and is expressed in terms of other root systems; see [CM, Theorem 1.1(1)]. However, there are other statements that are specific to the adjoint representation and yet can be obtained from our results in previous sections via Proposition 8.2. We now provide an alternate proof of one such statement from [CM].

Corollary 8.6. *Suppose \mathfrak{g} is simple with highest root θ , $\mathbb{V}^\lambda = \mathfrak{g} = L(\theta)$, and $i \in I \supset J$. Then the coordinate face $F_i = \text{conv}_{\mathbb{R}} \text{wt}_{I \setminus \{i\}} \mathfrak{g}$ is a codimension-one facet of the root polytope $\text{conv}_{\mathbb{R}} \text{wt} \mathfrak{g}$, if and only if $\widehat{I} \setminus \{\alpha_i\}$ is connected (i.e., the corresponding parabolic root subsystem is irreducible).*

Note that the result is precisely [CM, Theorem 4.5 (1) \Leftrightarrow (2)]. We now show that it quickly follows from the analysis in previous sections, via Proposition 8.2.

Proof. Note by Proposition 6.3 that $(I \setminus \{i\})_{\max} = I \setminus \{i\}$ for all $i \in I$. Now apply the equivalence (1) \Leftrightarrow (4) of Proposition 6.1 for $\mathbb{V}^\lambda = \mathfrak{g}$ simple). Thus, F_i is a codimension-one facet if and only if $(I \setminus \{i\})_{\min} = I \setminus \{i\}$, if and only if $\overline{\{i\}} = \{i\}$ by Proposition 8.2. It is easy to see using equation (8.4) that this condition is equivalent to $\widehat{I} \setminus \{\alpha_i\}$ being connected. \square

Remark 8.7. We make two further observations related to the recent paper [CM]. Recall that Proposition 3.10 provided a complete description of the inclusion relations among the sets $\text{wt}_J \mathbb{V}^\lambda$ (or their convex hulls) for any highest weight module \mathbb{V}^λ :

$$\begin{aligned} \text{wt}_J \mathbb{V}^\lambda \subset \text{wt}_{J'} \mathbb{V}^\lambda &\iff J_{\min} \subset J'_{\min}, J \setminus J(\mathbb{V}^\lambda) \subset J' \setminus J(\mathbb{V}^\lambda) \\ &\iff J_{\max} \subset J'_{\max}, J \setminus J(\mathbb{V}^\lambda) \subset J' \setminus J(\mathbb{V}^\lambda). \end{aligned}$$

Such a formulation (via the dictionary mentioned in Proposition 8.2) was also discussed in the special case of the finite-dimensional adjoint representation for simple \mathfrak{g} in [CM, Remark 4.6].

Next, for completeness we recall an interesting result shown recently by Cellini and Marietti for the adjoint representation of a simple Lie algebra. Namely, the authors show in [CM, Theorem 5.2] that no standard parabolic subset of $\text{wt} \mathfrak{g}$ is the union of two nontrivial orthogonal subsets. It is natural to ask if this result holds for other highest weight modules. However, this is not the case; in fact the result fails to hold even if \mathfrak{g} is simple and \mathbb{V}^λ is finite-dimensional. For example, when \mathfrak{g} is of type C_2 and $\lambda = \theta_s$, the highest short root, the (nonzero) weights of $L(\theta_s)$ comprise a root system of type $A_1 \times A_1$, hence can be partitioned into two nontrivial orthogonal subsets.

We end by pointing out that several of the main results in [CM] hold not only for the root polytope, but also for a large family of Weyl polytopes:

Proposition 8.8. *(Notation as in Theorem 2.8 and Definition 3.1.) Suppose $\text{supp}(\lambda) = \text{supp}(\theta)$ for some $\lambda \in P^+$. Define $F_J := \text{conv}_{\mathbb{R}} \text{wt}_{I \setminus J} L(\lambda)$ for $J \subset I$. Then [CM, Theorems 1.2, 1.3] hold for $\text{conv}_{\mathbb{R}}(\text{wt } L(\lambda))$, with the exact same formulas.*

One can similarly show that many of the other results in [CM] also go through completely unchanged for $\text{conv}_{\mathbb{R}}(\text{wt } L(\lambda))$, if λ and θ have the same support among the fundamental weights Ω .

Proof. The result follows from Theorem B, Lemma 5.2, Proposition 8.2, and Corollary 8.6 if we show that equation (8.3) holds for all $J \subset I$, for the simple finite-dimensional module $\mathbb{V}^\lambda = L(\lambda)$. But this is clear by Theorem A: the formulas in equations (3.2) and (3.3) for J_{\min}, J_{\max} only depend on $\text{supp}(\lambda)$ as well as J and $J \setminus J(\mathbb{V}^\lambda) = \emptyset$. \square

8.1. Finite-dimensional modules. Since the present paper was submitted, the work [LCL] by Li–Cao–Li has subsequently appeared. In it, the authors have independently extended Cellini–Marietti’s results in [CM] (by different methods than our analysis in the present paper) to all finite-dimensional simple modules over a simple Lie algebra \mathfrak{g} . As our analysis holds more generally for all highest weight modules and over all semisimple \mathfrak{g} , we conclude this section by discussing the main results of [LCL], and how they fit into our framework in a manner similar to the results in [CM].

In order to make the subsequent discussion more consistent with the analysis both in [CM] as well as above in this section, we begin by setting some notation. Fix a simple Lie algebra \mathfrak{g} and a dominant integral weight $\lambda \in P^+ \setminus \{0\}$. In [LCL], the authors define the *extended Coxeter diagram* by adding a new node $-\lambda$ to the Dynkin diagram of \mathfrak{g} , with a (single additional) edge between $-\lambda$ and a previous node $i \in I$ if $(\lambda, \alpha_i) > 0$. Now set $\widehat{I}_\lambda := I \cup \{-\lambda\}$, and for $J \subset I$, define $(\widehat{I}_\lambda \setminus J)_0$ to be the connected component of the extended Dynkin diagram of $\widehat{I}_\lambda \setminus J$ that contains the new node $-\lambda$. Finally, define $\overline{J} := I \setminus (\widehat{I}_\lambda \setminus J)_0$.

In [LCL], the authors study the Weyl polytope $\mathcal{P}(\lambda) = \text{conv}_{\mathbb{R}} \text{wt } L(\lambda)$, focusing on its “standard parabolic faces” $F_J^\lambda := \text{conv}_{\mathbb{R}} \text{wt}_{I \setminus J} L(\lambda)$. In the above notation, their main results are as follows:

- There exists a bijection between the distinct elements in the multiset $\{\text{conv}_{\mathbb{R}} \text{wt}_J L(\lambda) : J \subset I\}$ and the connected sub-diagrams of the extended Coxeter diagram on \widehat{I}_λ that contain the node $-\lambda$. The authors observe in [LCL] that this bijection is a special case of the *Putcha–Renner Recipe* [PR, Theorem 4.16], which yields connections to the theory of algebraic monoids.
- Each face F_J^λ has affine hull $\mathbb{R}\Delta_{I \setminus \overline{J}} = \mathbb{R}\Delta_{(\widehat{I}_\lambda \setminus J)_0 \setminus \{-\lambda\}}$, which is spanned by roots. Moreover, $\dim F_J^\lambda = |(\widehat{I}_\lambda \setminus J)_0| - 1$.
- Given $J, J' \subset I$, $F_J^\lambda = F_{J'}^\lambda$ if and only if $(\widehat{I}_\lambda \setminus J)_0 = (\widehat{I}_\lambda \setminus J')_0$, or equivalently, if $\overline{J} = \overline{J'}$.
- The barycenter of the face F_J^λ is a nonnegative rational linear combination of the fundamental weights in Ω_J .

- The f -polynomial of the Weyl polytope $\text{conv}_{\mathbb{R}} \text{wt } L(\lambda)$ is

$$\mathbf{f}_{L(\lambda)}(t) = \sum_J [W : W_{(I \setminus J) \cup (\widehat{I}_\lambda \setminus J)^\perp}] t^{|\widehat{I}_\lambda \setminus J| - 1},$$

where the sum runs over all $J \subset I$ such that $\widehat{I}_\lambda \setminus J$ is a connected sub-diagram of the extended Coxeter diagram.

Note that when $\lambda = \theta$ is the highest root of \mathfrak{g} , the corresponding extended Coxeter diagram on \widehat{I}_θ is precisely (the simple graph underlying) the affine Dynkin diagram corresponding to \mathfrak{g} , and in this situation the above results were shown in [CM]. Similarly, the case where $\lambda \in P^+$ has the same support as θ was previously worked out in Proposition 8.8. For other $\lambda \in P^+$, the results in [LCL] are formulated along the lines of those in [CM], and extend many of the results in [CM].

We now explain why for all $\lambda \in P^+$, the results shown in [LCL] also follow from the above analysis in the present paper, if the same dictionary as in Proposition 8.2 is used. To do so, one first needs to define the analogue of the set ∂J of simple roots in [CM], which the authors do not define for general $\lambda \in P^+$ in [LCL]. Thus, we define (via Definition 3.1(5))

$$(8.9) \quad \partial J := J \setminus (\widehat{I}_\lambda \setminus J)_0^\perp \subset I.$$

Now observe that Proposition 8.2 (with the new definitions of \overline{J} and ∂J) holds for $\mathbb{V}^\lambda = L(\lambda)$ for all $\lambda \in P^+$, with the proof essentially unchanged. Moreover, the formulas for J_{\min}, J_{\max} depend only on λ through its support, since one still has $J(\mathbb{V}^\lambda) = J(L(\lambda)) = I$. It follows that the results in [LCL] (and others, including analogues of the results in [CM] and our main results in Section 3) hold in $\text{wt } L(\lambda)$ for all $\lambda \in P^+$.

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